## Chapter 5

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# THEORETICAL EXPLANATION OF FATIGUE MECHANISM IN CLAMPED TENSION COMPONENTS

**Abstract:** The paper presents a theoretical explanation of the phenomena observed in tension components subjected to bending fatigue. It explains why this fatigue occurs close to anchorages and it intensity increases with value of tensile force in a cable. The *Mathematica* system has been used to solve the problem and illustrates the solution graphically.

**Key words:** Fatigue, tension components, anchorage, cable, rope, *Mathematica*, differential equation, second order theory

## X.1. Introduction

There are two types of fatigue in tension components: due to axial and bending behavior [2]. This article focuses only on the second one.

Many of articles, design standards and books on vibrations and fatigue [1,2,6,7,8,9] of structures with tension components like ropes and cables report two crucial observations:

- the fatigue is observed mainly near the connections (anchorages),
- the fatigue intensity enlarges with value of tensile axial force in the component.

Eurocode 3 part 1-11 [9] says: *Fatigue failure of cable systems usually occurs at anchorages, saddles or clamps*.

Procedures of cable testing for bridges are described in [3, 4]. The testing procedures simulate behavior of cables in real structures and consist in transverse and rotational deformations of clamped on both ends ~~of~~ prestressed ropes.

There is shown that these experiments and practical observations can be mathematical explained. To carry out this Wolfram *Mathematica* 10.2 www.wolfram.com system has been used to solve the problem of a bent bar subjected to large tensile axial force. The theoretical basis of the plot is described in [5]. This paper focuses on merits, only.

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## X.2. Problem formulation

In real structures with tensioned components, such elements behave as clamped near the anchorage [2]. Therefore, deformation of the element subjected to axial tensile force and two types of kinematic enforcements producing bending in them are considered. The scheme of the considered element is shown in Fig. 1.

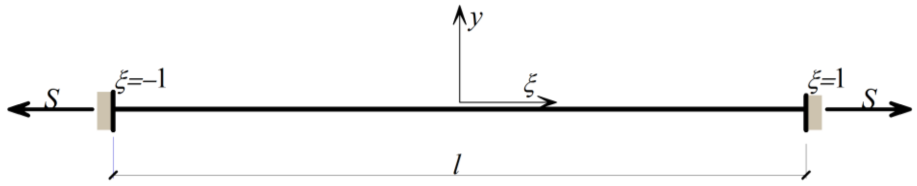


***Fig. 1. Statical scheme of a clamped-clamped bar subjected to tension force***

Following the [5], a dimensionless coordinate $\xi$ which is defined with (X.1) is introduced, where $x$ is a physical coordinate measured along the bar. The beginning of the coordinate system is in the middle of the beam.

$$\xi = \frac{2\,x}{l}, \quad -1 \le \xi \le 1. \tag{X.1}$$

Due to change of variables the following rule of differentiation holds:

$$\partial_{\{x,n\}} f\left(x\right) = \frac{\partial^n f\left(x\right)}{\partial x^n} = \frac{2^n}{l^n} \partial_{\{\xi,n\}} f\left(\xi\right) = \frac{2^n}{l^n} \frac{\partial^n f\left(\xi\right)}{\partial \xi^n}. \tag{X.2}$$

The considered problem is described with the following homogenous differential equation of order two of the function of transverse displacement $w(\xi)$:

$$w^{(4)} - \lambda^2 w'' = 0, \tag{X.3}$$

where parameter $\lambda$:

$$\lambda = \frac{l}{2}\sqrt{\frac{S}{EJ}}. \tag{X.4}$$

Here $S$ is an axial tensile force in a bar, $l$ is its length, $EJ$ is its bending stiffness. If $S = 0$ the equation describes the first order theory problem of bending of a straight bar.

## X.3. Problem solution

As it has been mentioned the problem is solved, and illustrated with aid of *Mathematica* 10.2 system. To solve differential equation **DSolve** function is applied. **ExpToTrig** converts exponentials to trigonometric functions. **FullSimplify** tries a wide range of transformations involving elementary and special functions and returns the simplest form it finds. **Assuming** helps to find simplest form taking into account given assumptions.

Further description of the used functions can be found in the system documentation or Wolfram Language & System Documentation Center [http://reference.wolfram.com/language/](http://reference.wolfram.com/language/) .

We will consider two cases: transverse (lateral) displacement of clamps and rotation of them. The influence of axial tensile force on the bar behaviour is considered.

### X.3.1. Transverse displacement of clamps

Transverse displacement of the right clamp is described with following boundary conditions: $w(-1)=0, w'(-1)=0, w(1)=1, w'(1)=0$ . The set of *Mathematica* functions produces the solution in a form of hyperbolic trigonometric functions.

$$\mathbf{Assuming}[\xi < 1\&\&\xi > -1, \mathbf{FullSimplify}[\mathbf{ExpToTrig}[\mathbf{DSolve}[\{w^{(4)}[\xi] - \lambda^2 w''[\xi] == 0, w[-1] == 0, w[1] == 1, w'[-1] == 0, w'[1] == 0\}, w[\xi], \xi]]]]]$$

$$\{\{w[\xi] \to -\frac{-\lambda(1+\xi)\mathrm{Cosh}[\lambda] + \mathrm{Sinh}[\lambda] + \mathrm{Sinh}[\lambda\xi]}{2\lambda\mathrm{Cosh}[\lambda] - 2\mathrm{Sinh}[\lambda]}\}\}$$

When $\lambda \to 0$ the solution is a well known function of first order theory of Structural Mechanics describing deformation of clamped-clamped bar subjected to transverse deformation of a bar.

$$\mathbf{Limit}[-\frac{-\lambda(1+\xi)\mathbf{Cosh}[\lambda] + \mathbf{Sinh}[\lambda] + \mathbf{Sinh}[\lambda\xi]}{2\lambda\mathbf{Cosh}[\lambda] - 2\mathbf{Sinh}[\lambda]}, \lambda \to 0]$$

$$-\frac{1}{4}(-2+\xi)(1+\xi)^2$$

The deformation of the bar for $\lambda = 0$ is show in Fig. 2.

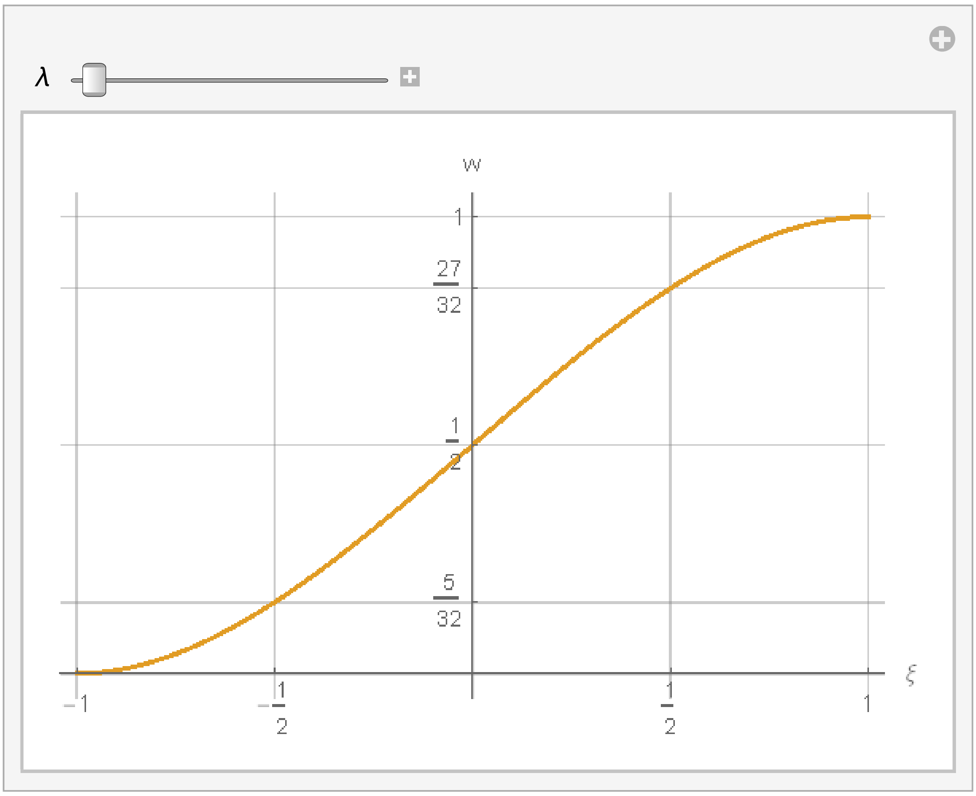


***Fig. 2. Deformation of the bar subjected to the unit transverse displacement of the right support, for*** $\lambda = 0$

Readers familiar with Finite Element Method will find it as a very popular shape function.

The case of the bar subjected to tensile axial force $S > 0$ is presented in Fig. 3. It can be seen that in the middle of the span the bar straightens and near clamps intensively bends.

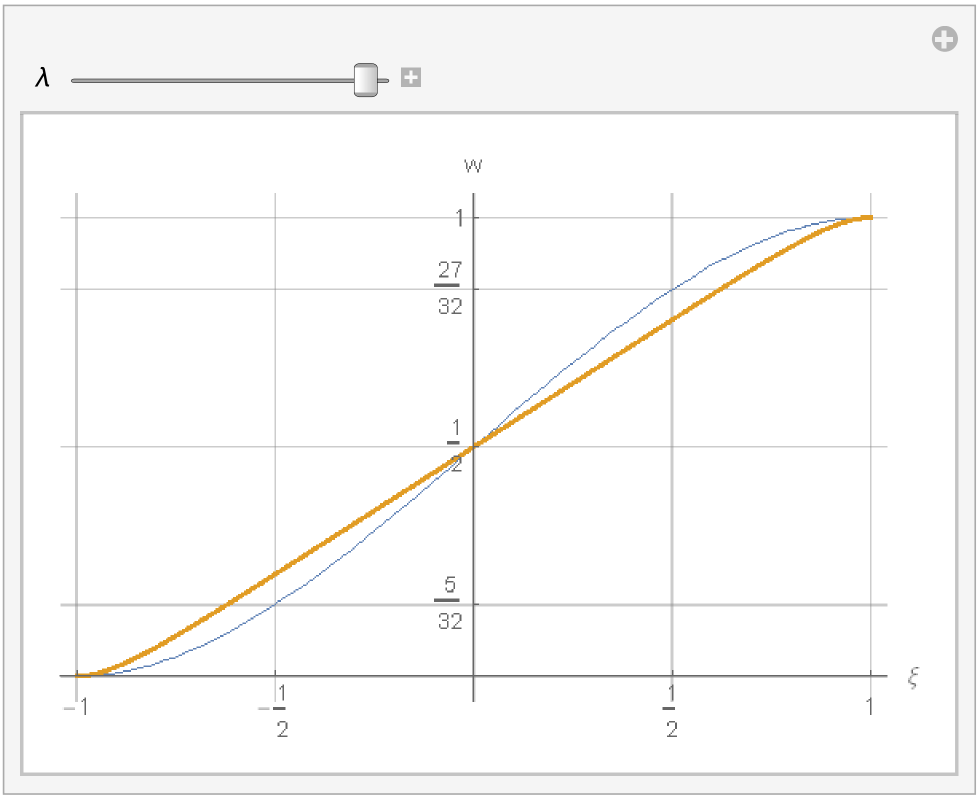


***Fig. 3. Deformation of the bar subjected to the unit transverse displacement of the right support, for*** $\lambda > 0$

According to rule (X.2) the function of bending moment as a function of $\xi$ is defined with the following function.

$$M = -4\, EJ \frac{w''}{l^2} \qquad \text{(X.5)}$$

We can evaluate the moment function both for $\lambda = 0$ and $\lambda > 0$ applying it at once for both above expressions.

$$\textbf{Assuming}[\xi < 1\&\&\xi > -1, \textbf{FullSimplify}[-4\partial_{\{\xi,2\}}\{-\frac{1}{4}(-2+\xi)(1+\xi)^2, -\frac{\textbf{Sinh}[\lambda\xi] - \lambda(\xi+1)\textbf{Cosh}[\lambda] + \textbf{Sinh}[\lambda]}{2\lambda\textbf{Cosh}[\lambda] - 2\textbf{Sinh}[\lambda]}\}]]$$

$$\{6\xi, \frac{2\lambda^2\text{Sinh}[\lambda\xi]}{\lambda\text{Cosh}[\lambda] - \text{Sinh}[\lambda]}\}$$

Funtion of bending moment for a bar not subjected to axial force is presented in Fig. 4.

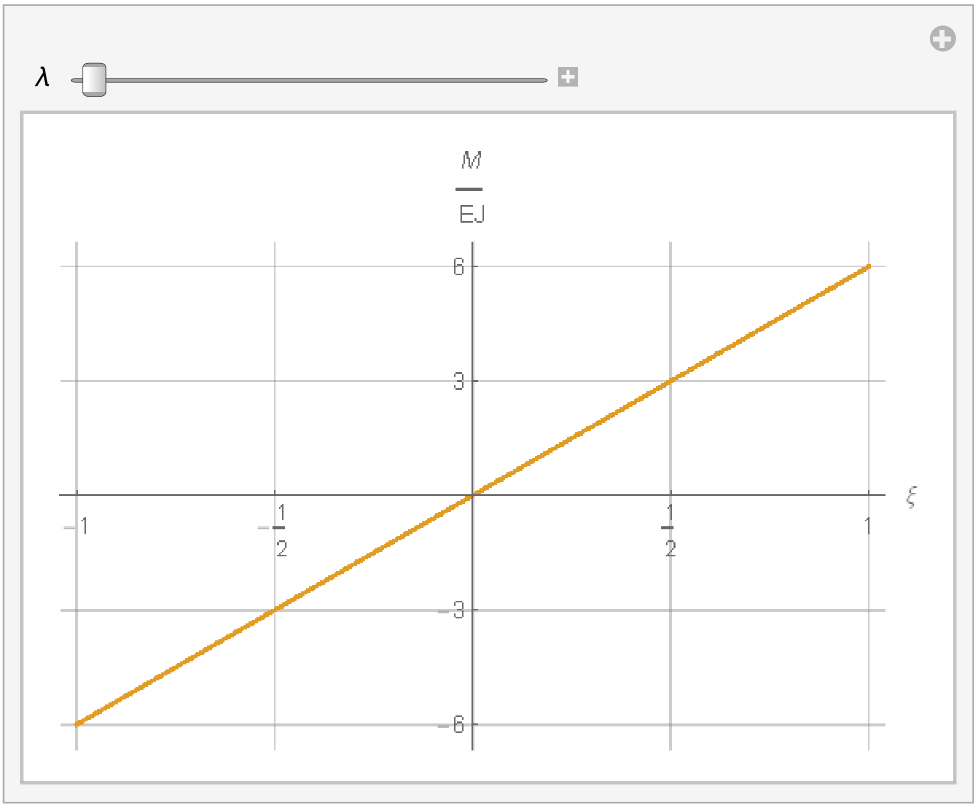


*Fig. 4. Bending moment of the bar subjected to the unit transverse displacement of the right support, for* $\lambda = 0$

Function of bending moment for a bar under a tensile axial force is presented in Fig. 5. A thin line represents the first order theory solution.

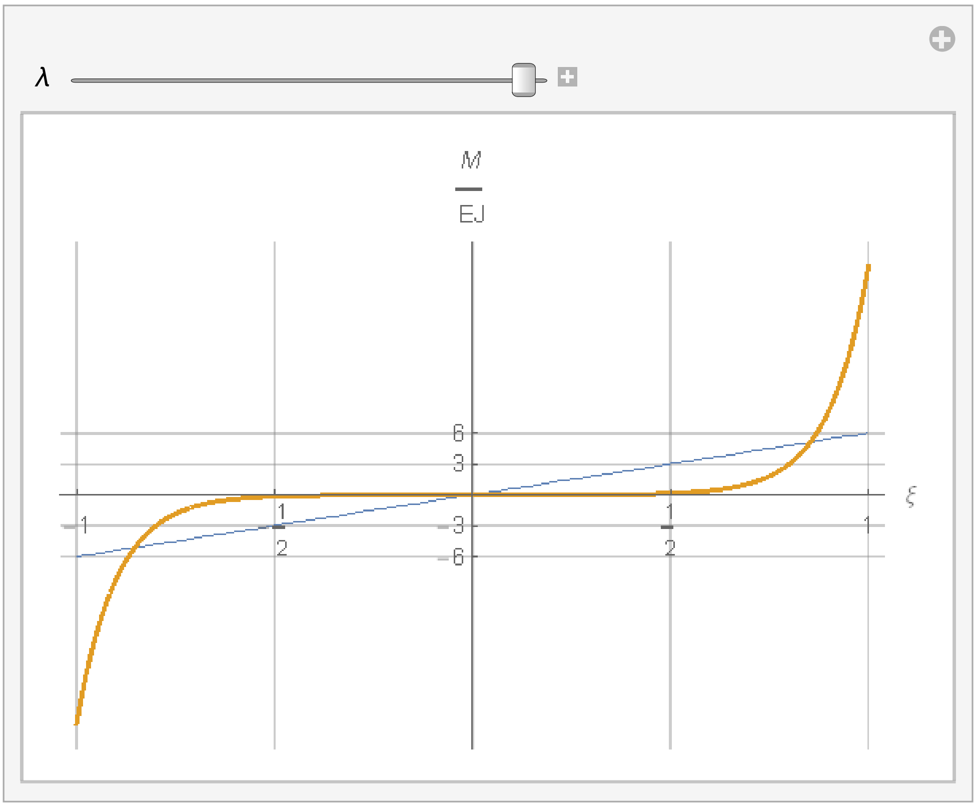


***Fig. 5. Bending moment of the bar subjected to the unit transverse displacement of the right support, for*** $\lambda = 0$

It can be observed that in the middle of the span bending moment quickly decays to zero. Closer to anchorages it firstly increases and next goes to zero. Only for $\xi = -1$ and $\xi = 1$ (in anchorages) it aims uniforrmly to infinity. Figure 6 shows the bending moment for $\xi = 1$ as a function of parameter $\lambda$. It can be seen that this function has a oblique asymptote:

$$f(\lambda) := 2EJ\,(1+\lambda) \tag{X.6}$$

so it can be concluded that the bending moment near clamps increases asymptotically to the $\sqrt{S}$ , see (X.4).

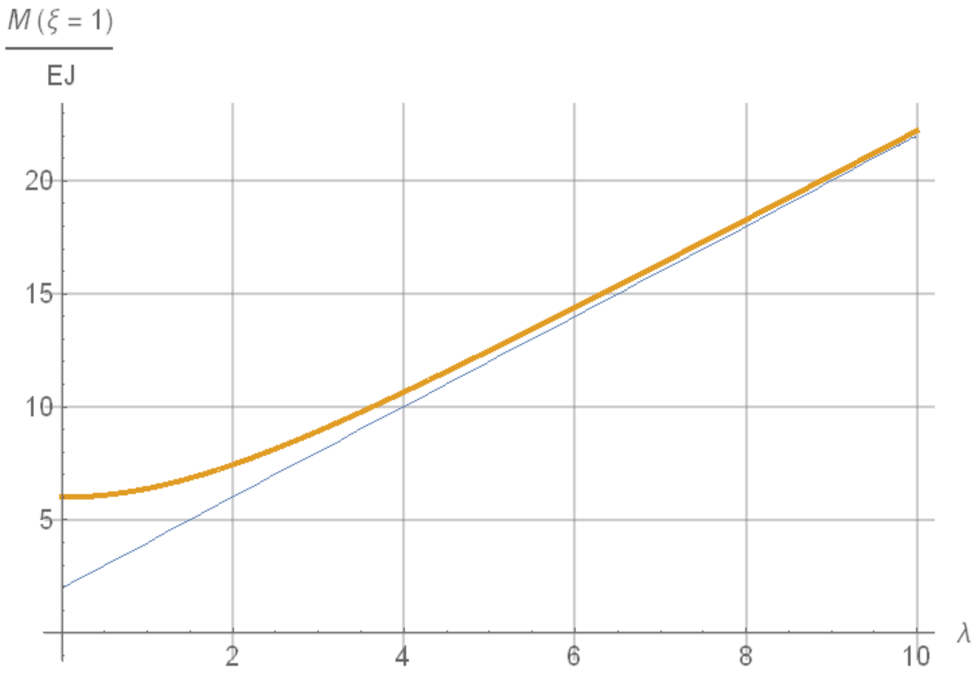


***Fig. 6. Bending moment by the right clamp of the bar subjected to the unit transverse displacement as function of parameter*** $\lambda$

By this, both observations about tension components fatigue are explained:

– the fatigue is observed mainly near the connections (anchorages), because axial force in the tension components reduces moments in the middle of the span and increases it at the ends,
– the fatigue intensity enlarges with value of tensile axial force in the component, because the above mentioned increase of the bending moment close to the anchorages depends on axial force in it.

### X.3.2. Rotation of clamps

Similar analysis can be done with regard to rotation of clamps. Unit rotation of the right clamp is described with following boundary conditions, (taking into account X.2): $w(-1)=0,\ w'(-1)=0,\ w(1)=0,\ w'(1)=-\frac{1}{2}$ . The following set of *Mathematica* functions produces the solution:

**Assuming**$\left[\xi < 1\&\&\xi > -1\right.$

$> -1$, **FullSimplify**[**ExpToTrig**[**DSolve**[{$w^{(4)}[\xi] - \lambda^2 w''[\xi]$ $== 0, w[-1] == 0, w[1] == 0, w'[-1] == 0, w'[1] =$ $= -\frac{1}{2}$}, $w[\xi], \xi$]]]]

$$\{\{w[\xi] \to \frac{1}{8\lambda(-1 + \lambda\text{Coth}[\lambda])}\text{Csch}[\lambda](-2\text{Cosh}[\lambda] + 2\text{Cosh}[\lambda\xi] + \lambda(1 - \xi + (1 + \xi)\text{Cosh}[2\lambda] - 2\text{Cosh}[\lambda + \lambda\xi])\text{Csch}[\lambda])\}\}$$

The diagram of this displacement function for $\lambda = 0$ is shown in Fig. 7. This function is described with formula: $w_o = \frac{1}{8}(1-\xi)(1+\xi)^2$.

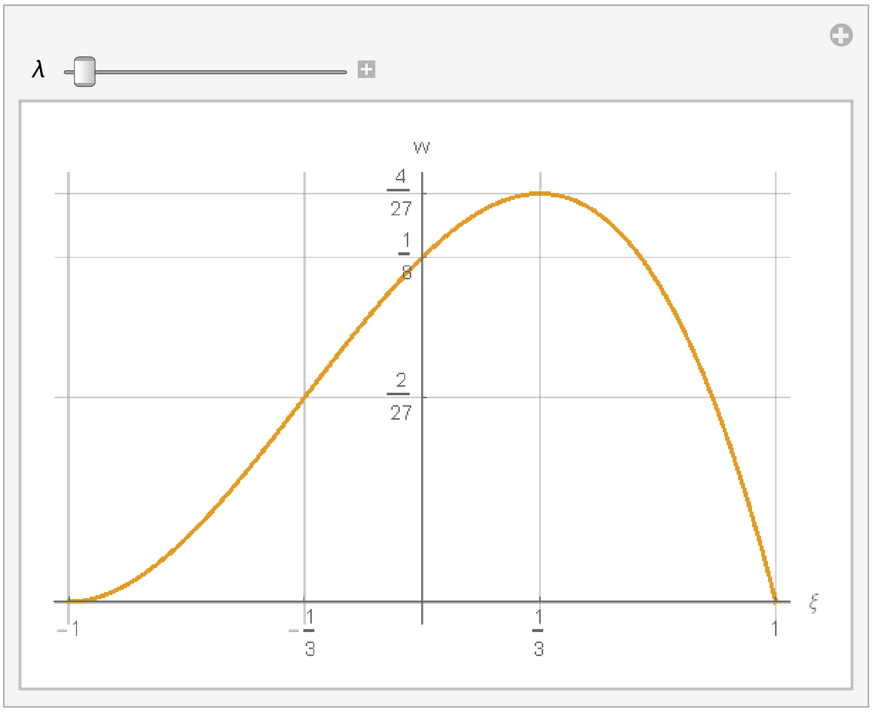


*Fig. 7. Deformation of the bar subjected to the unit rotation of the right support, for* $\lambda = 0$

Similarly to the previous example in case of the bar subjected to tensile axial force (Fig. 8) in the middle of the span the bar straightens and near right clamp intensively bends.

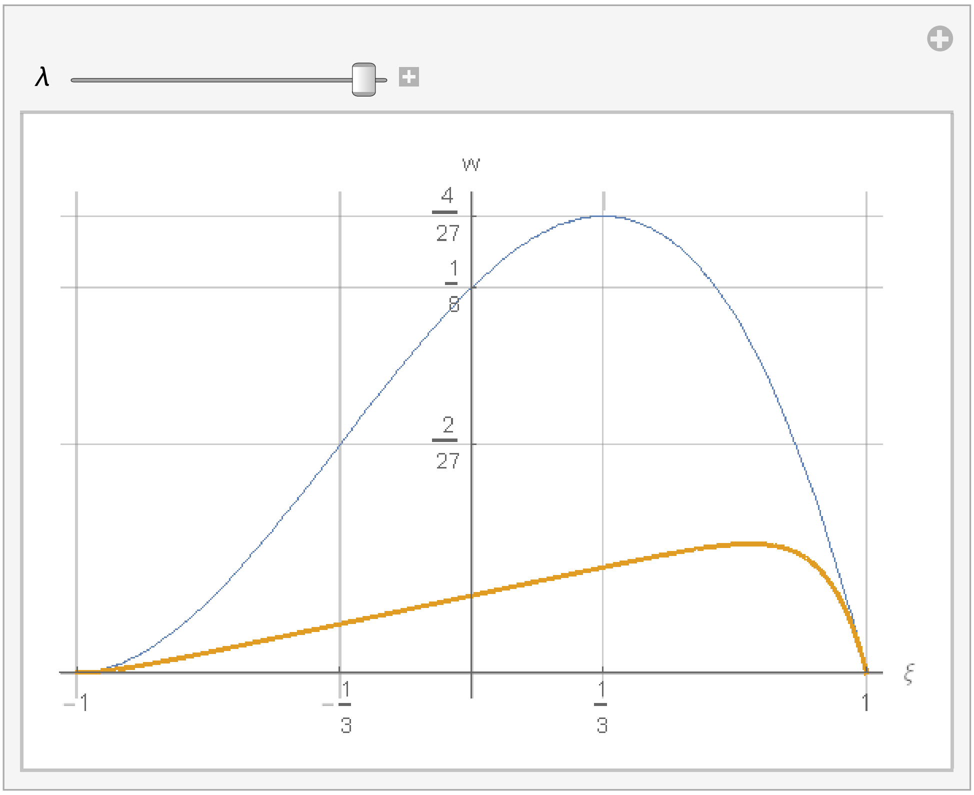


***Fig. 8. Deformation of the bar subjected to the unit rotation of the right support, for*** $\lambda > 0$

The moment function both for $\lambda = 0$ and $\lambda > 0$ applying the formula for bending moment can be evaluated.

$$\mathbf{Assuming}[\xi < 1\&\&\xi > -1, \mathbf{FullSimplify}[-\frac{\mathbf{4EJ}}{\boldsymbol{l^2}}\partial_{\{\xi,2\}}\{-\frac{\mathbf{1}}{\mathbf{8}}(-\mathbf{1} + \xi)(\mathbf{1}+\xi)^2, \frac{\mathbf{1}}{\mathbf{8}\lambda(-\mathbf{1}+\lambda\mathbf{Coth}[\lambda])}\mathbf{Csch}[\lambda](-\mathbf{2Cosh}[\lambda] + \mathbf{2Cosh}[\lambda\xi] + \lambda(\mathbf{1} - \xi + (\mathbf{1}+\xi)\mathbf{Cosh}[\mathbf{2}\lambda] - \mathbf{2Cosh}[\lambda + \lambda\xi])\mathbf{Csch}[\lambda])\}]]$$

$$\{\frac{\mathrm{EJ} + 3\mathrm{EJ}\xi}{l^2}, \frac{\mathrm{EJ}\lambda\mathrm{Csch}[\lambda](-\mathrm{Cosh}[\lambda\xi] + \lambda\mathrm{Cosh}[\lambda+\lambda\xi]\mathrm{Csch}[\lambda])}{l^2(-1+\lambda\mathrm{Coth}[\lambda])}\}$$

The diagram of the first case, $\lambda = 0$, is shown below.

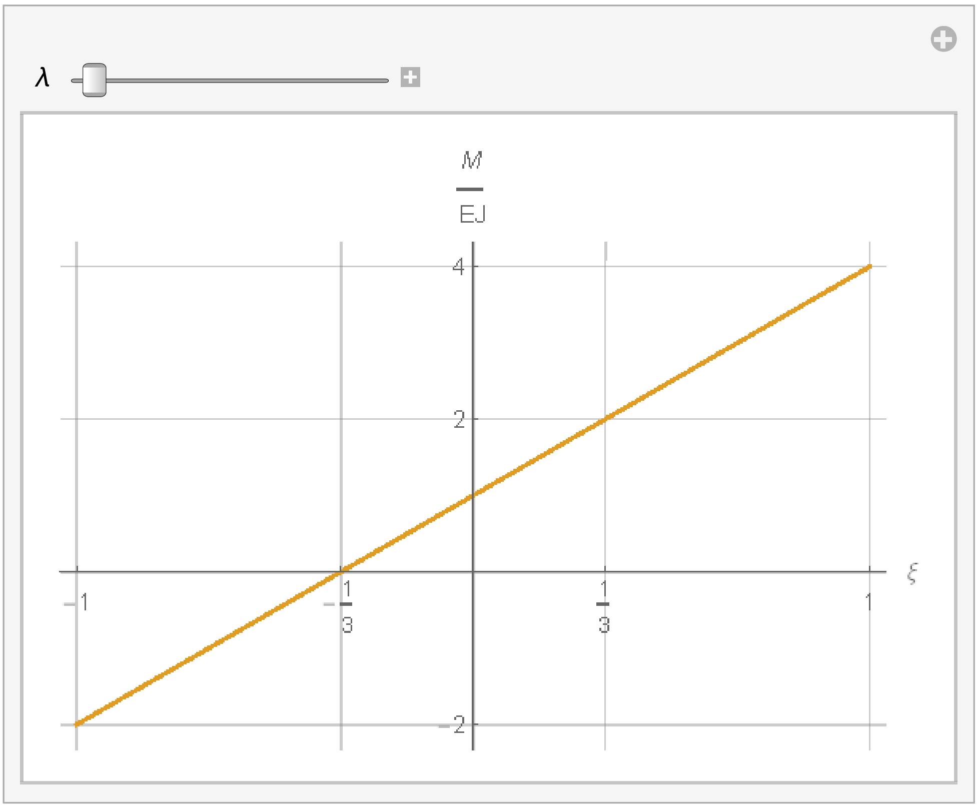


***Fig. 9. Bending moment of the bar subjected to the unit rotation of the right support, for*** $\lambda = 0$

Function of bending moment for a bar under a tensile axial force is presented in Fig. 10. A thin line represents the first order theory solution.

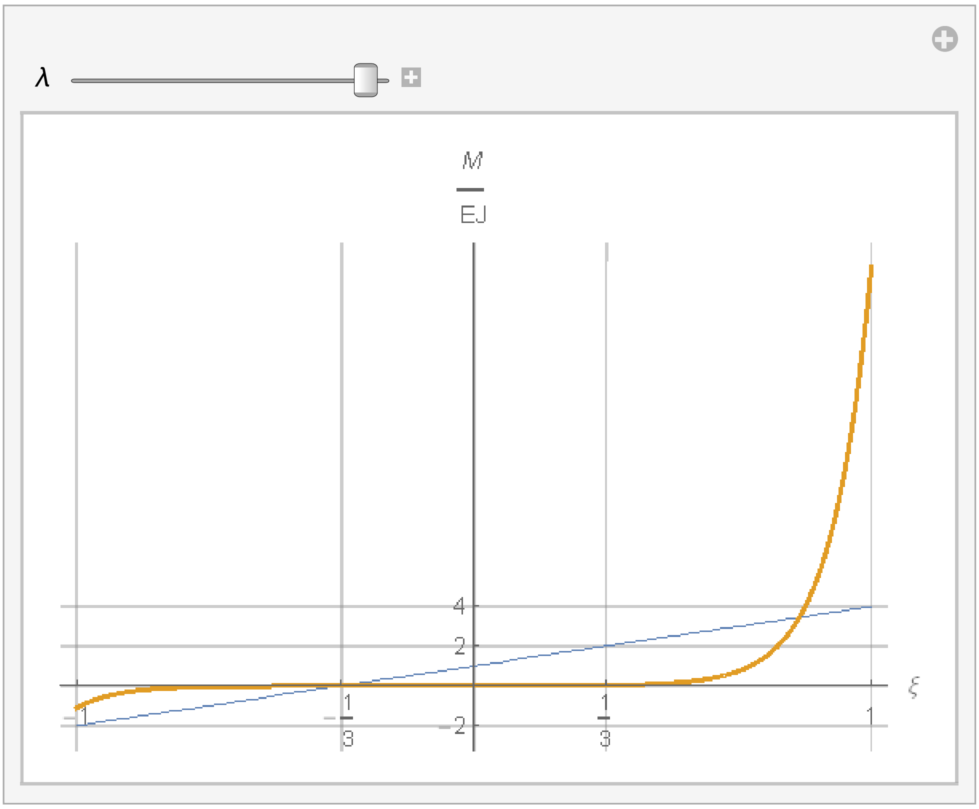


***Fig. 10. Bending moment of the bar subjected to the unit rotation of the right support, for*** $\lambda > 0$

It can be observed that in the middle of the span bending moment quickly decays to zero. In the left anchorage, $\xi = -1$, it decays to 1. Close to the right abchorage it first increase and next decay to zero. Only for $\xi = 1$ (right anchorage) it aims uniforrmly to infinity. Figure 11 shows the bending moment for $\xi = 1$ as a function of parameter $\lambda$.

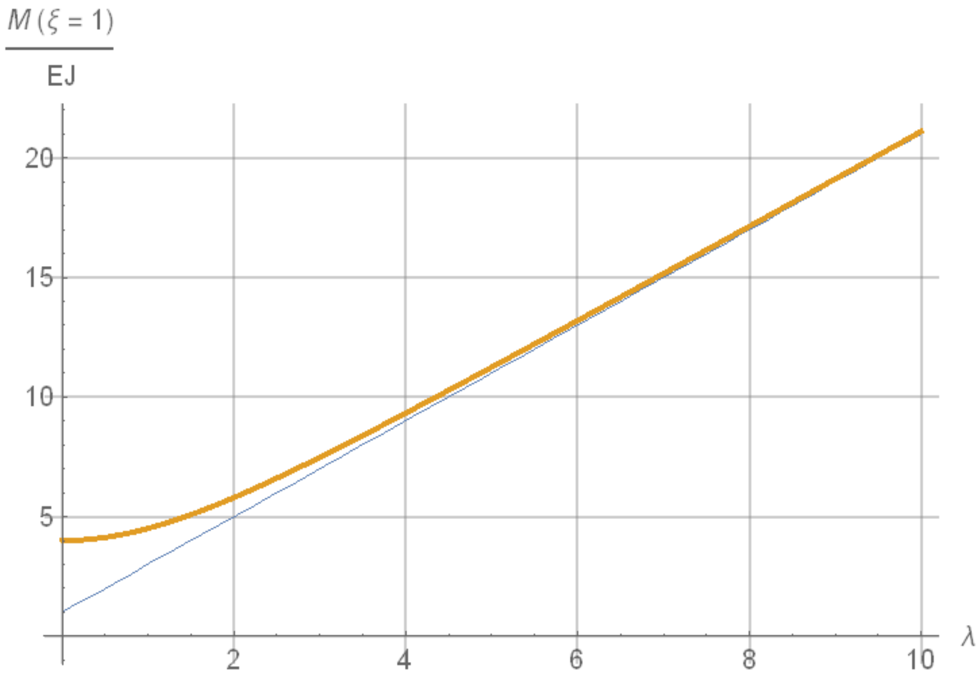


***Fig. 11. Bending moment by the right clamp of the bar subjected to the unit rotation as function of parameter $\lambda$***

It can be seen that this function has a oblique asymptote:

$$g(\lambda) := EJ\ (1+2\lambda) \qquad \text{(X.6)}$$

so also in this case it can be concluded the bending moment near rotated clamp increases with asymptotical proportion to the $\sqrt{S}$ , see (X.4).

Concluding this example, explanations of the observed phenomena carried out in the previous subsection can be confirmed.

## X.4. Conclusions

The axial forces in tension components (ropes, cables) have a crucial influence on fatigue of these elements. This paper explains theoretically the main reasons of observed phenomena.

By this, both observations about tension components fatigue are explained:

– since axial force in the tension components reduces moments in the middle of the span and increases it at the ends - the fatigue is observed mainly near the connections (anchorages),

– since increase of bending moment close to the anchorages depends on asymptotical proportion to the square root of tensile axial force - the fatigue intensity enlarges with value of this force in the component.

The above mentioned fact has a crucial influence on safety of structures with tension components. Tension components are usually tested according to technical specifications and design standards (including Eurocode) for bending fatigue under the axial force not bigger than 0,45 of their bearing capacity. Overriding of this value in structures will lead to significant reduction of expected durability of the structure since larger axial forces produce larger bending moments near clamps.

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