11. E. Cartan, Exterior Differential Systems and Their Geometric Applications [Russian translation], Izd. Mosk. Gos. Univ., Moscow (1962).
12. S. P. Finikov, The Method of the Cartan Exterior Forms in Differential Geometry [in Russian], Ob''edin. Gos. Izdat., Moscow–Leningrad (1948).

THE ANDREWS–CURTIS PROBLEM FOR $\mathcal{F}(\mathfrak{M})$

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We show that if the problem, mentioned in the title, has an affirmative solution in a variety $\mathfrak{M}$, then it has an affirmative solution also in the variety $\mathfrak{M}^k$ (k ≥ 1) and in all intermediate varieties. Hence it follows immediately that the problem has affirmative solution in the variety of solvable groups [1].

Let G be an arbitrary group. We will consider the set of all n-tuples $(a_1, \ldots, a_n)$, n ≥ 2, of elements of G. By definition, such an n-tuple of elements is called a generating n-tuple if its components generate G and is said to be degenerate if the normal divisor, generated by its components, coincides with G.

The following transformations on the set of these n-tuples are called elementary Nielsen transformations:

1) $(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) \to (a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n)$;

2) $(a_1, \ldots, a_i, \ldots, a_n) \to (a_1, \ldots, a_i^{-1}, \ldots, a_n)$;

3) $(a_1, \ldots, a_i, \ldots, a_n) \to (a_1, \ldots, a_i a_j, \ldots, a_n)$, $i \neq j$.

These elementary transformations generate the group N of the Nielsen transformations, which is anti-isomorphic with the group $\mathrm{Aut}\, F_n$ of automorphisms of the free group of rank n. If G is the Hopf relatively free group of rank n, then it is easily seen that N acts transitively on the set of generating n-tuples if and only if the natural mapping $\mathrm{Aut}\, F \to \mathrm{Aut}\, G$ is an epimorphism. This holds if G is free (free Abelian, metabelian of rank not equal to e [2, 3], or free nilpotent of class at most 2 [4]). No other examples are known.

In algebraic topology, Andrews and Curtis have supplemented the set of elementary transformations on n-tuple of words, introducing transformation of a single component by an arbitrary element of the group:

4) $(a_1, \ldots, a_i, \ldots, a_n) \to (a_1, \ldots, a_i^d, \ldots, a_n)$.

The transformations 1) - 4) generate the group T of extended Nielsen transformations. The set of degenerate n-tuples of elements is closed with respect to the action of T. The problem of transitivity of action of T coincides with the well-known Andrews–Curtis problem if G is a free group.

Let F be a free group with a basis $x_1, \ldots, x_n$ and H be a normal divisor of it that is contained in the commutant. Let V denote the verbal subgroup generated in F by the words from H and let V(H) denote the corresponding verbal subgroup in H. We will denote the equivalence of two n-tuples of elements with respect to T by ~ and the equivalence with respect to N by ≃. Without loss of generality, we take n = 2 in the following discussions for simplicity.

If a 2-tuple has the form (ab, c), then the transformation of the first component by the element b^{-1} gives the equivalence $(ab, c) \sim (ba, c)$. Combining this with an elementary transformation of the form 3), we get the equivalence

$$(ac^{-1}b, cd) \sim (adb, cd), \ (acb, cd) \sim (ad^{-1}b, cd).$$

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Hence the following property follows easily:

Property 1. $(x_1c_1, x_2c_2) \sim (x_1c_1', x_2c_2)$, where c_1' is obtained from the word c_1 by replacing x_2 by c_2^{-1} in its expression, i.e., $c_1 = c_1(x_1, x_2)$, $c_1' = c_1(x_1, c_2^{-1})$.

LEMMA 1. Each n-tuple of the form $(x_1h_1, \ldots, x_nh_n)$, $h_i \in H$, is equivalent with respect to the action of T to the n-tuple $(x_1h_1', \ldots, x_nh_n')$, where $h_1' \in V(H)$.

Proof. Using Property 1 repeatedly, we can replace each x_i $(i \neq 1)$ in the expression $h_1 = h_1(x_1, \ldots, x_n)$ by h_i^{-1}. If x_1 does not occur in the expression of h_1, then we get the desired $h_1' = h_1(h_2^{-1}, \ldots, h_n^{-1}) \in V(H)$. But if x_1 occurs in the expression of h_1, then we get $k_1 = h_1(x_1, h_2^{-1}, \ldots, h_n^{-1})$. Since $H \lhd F$ and $H \subseteq F'$, it follows that $k_1 \in H$. Further we show that we can pass from k_1 to $k_1' = h_1((x_2c)^{-1}, h_2^{-1}, \ldots, h_n^{-1})$ for a certain $c \in H$. Moreover, it follows from these properties of H that $k_1' \in H$. Finally, we get the desired h_1' by replacing x_2 by k_2^{-1} for a certain $k_2 \in H$, i.e., $h_1' = h_1(c^{-1}k_2, h_2^{-1}, \ldots, h_n^{-1}) \in V(H)$. The possibility of the indicated passage follows from the chain of equivalences

$$(x_1k_1, x_2h_2) \approx (x_1k_1, x_1k_1x_2h_2) = (x_1k_1, x_1x_2c) \sim (x_1k_1', x_1x_2c) \approx$$
$$\approx (x_1k_1', k_2'^{-1}x_2c) = (x_1k_1', x_2k_2) \sim (x_1h_1', x_2k_2), \text{ where } h_1' \in V(H),\ k_2 \in H.$$

The indicated transformations transform the n-tuple $(x_1h_1, x_2h_2, \ldots, x_nh_n)$ into the n-tuple $(x_1h_1', x_2k_2, x_3h_3, \ldots, x_nh_n)$. We make a cyclic permutation, putting x_1h_1' in the last place, and repeat the arguments. Thus, we arrive at the n-tuple $(x_{n-1}h_{n-1}', x_nk_n, x_1h_1', \ldots, x_{n-2}h_{n-2}')$, where $h_1' \in V(H)$, $k_n \in H$. Using Property 1, we replace k_n by $h_n' = k_n(h_1'^{-1}, \ldots, h_{n-1}'^{-1}, x_n)$. Since $k_n \in H \subseteq F'$ and $h_1' \in V(H)$, we have $h_n' \in V(H)$. The corresponding permutation now gives the desired n-tuple.

THEOREM 1. Let F be a free group with a basis $x_1, x_2, \ldots, x_n$ $(n \geq 2)$. Let $H \lhd F$, $H \subseteq F'$, and, moreover, $U \lhd F$, $H \supseteq U \supseteq V(H)$. If the group T of extended Nielsen transformations acts transitively in F/H, then T also acts transitively in F/U.

Proof. Let us consider the natural mappings $F \to F/V(H) \to F/U \to F/H$. Let $(\bar{a}_1, \ldots, \bar{a}_n)$ be a degenerate n-tuple of elements in F/U and $(\bar{\bar{a}}_1, \ldots, \bar{\bar{a}}_n)$ be its image in F/H. Then, by the condition, there exists a transformation T_1: $(\bar{\bar{a}}_1, \ldots, \bar{\bar{a}}_n) \to (\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_n)$. Let $(a_1, \ldots, a_n)$ be an arbitrary preimage of the given n-tuple in F. Then T_1: $(a_1, \ldots, a_n) \to (x_1h_1, \ldots, x_nh_n)$, $h_i \in H$. By Lemma 1, there exists a transformation T_2: $(x_1h_1, \ldots, x_nh_n) \to (x_1h_1', \ldots, x_nh_n')$, $h_i' \in V(H)$. Consequently, T_1T_2 transforms $(\bar{a}_1, \ldots, \bar{a}_n)$ into $(\bar{x}_1, \ldots, \bar{x}_n)$, which was desired to be proved.

Remark. The theorem from [1] is obtained for $U = [H, H]$.

Let $F_n(\mathfrak{M})$ denote the free group of the variety $\mathfrak{M}$ of the corresponding verbal subgroup, contained in the commutant.

We have proved the following theorem.

THEOREM 2. If the group T of extended Nielsen transformations acts transitively in $F_n(\mathfrak{M})$ on the set of degenerate n-tuples of elements, then it acts transitively also in $F_n(\mathfrak{M}^k)$ $(k \geq 1)$ and in the free groups of the intermediate varieties.

COROLLARY. In each free group $F_n(\mathfrak{A}^k)$ of solvable variety, the group T acts transitively on the set of degenerate n-tuples of elements.

LITERATURE CITED

1. A. G. Myasnikov, "Extended Nielsen transformations and triviality of a group," Mat. Zametki, 35, No. 4, 491-495 (1984).
2. S. Bachmuth and H. Y. Mochizuki, "Aut(F) → Aut(F/F") is surjective for free group F of rank ≥ 4," Trans. Am. Math. Soc., 292, 81-101 (1985).
3. V. A. Roman'kov, "Automorphism groups of free metabelian groups," in: Problems of Interconnection of Abstract and Applied Algebra [in Russian], Novosibirsk (1985); pp. 53-80.
4. S. Andreadakis, "On the automorphisms of free groups and free nilpotent groups," Proc. London Math. Soc., 15, 239-268 (1965).