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WHEN ENDOMORPHISMS OF G INDUCING AUTOMORPHISMS OF G/V ARE AUTOMORPHISMS

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1. Introduction

Let G denote a relatively free group of a finite or countably infinite rank with a fixed set of free generators $x_1, x_2, \ldots, G'$ the commutator subgroup, and V a verbal subgroup belonging to G'. Following H. Neumann [6] we shall use the vector representation for endomorphisms of G. Vector $\mathbf{v}=(v_1, v_2, \ldots)$ represents an endomorphism $\mathbf{v}$ such that $x_i\mathbf{v}=v_i$ for all i. The identity map is represented by $\mathbf{1}=(x_1, x_2, \ldots)$. We need also the trivial endomorphism $\mathbf{0}=(e, e, \ldots)$. The length of vectors is equal to the rank of G. We shall consider the near-ring of vectors, with addition and multiplication given below $\mathbf{u}+\mathbf{v}=(u_1v_1, u_2v_2, \ldots)$ where u_iv_i is a product in G, and $\mathbf{uv}=(u_1\mathbf{v}, u_2\mathbf{v}, \ldots)$ where $u_i\mathbf{v}$ is the image of u_i under the endomorphism $\mathbf{v}$. There is only one distributivity law $(\mathbf{u}+\mathbf{v})\mathbf{w}=\mathbf{uw}+\mathbf{vw}$.

If we denote by (V) the set of all vectors with components from V, then the set of all endomorphisms of G which induce the identity map in G/V must be denoted as $\mathbf{1}+(V)$. The question we are concerned with is when the natural map $\alpha:\operatorname{Aut} G \to \operatorname{Aut} G/V$ is onto. It is known [4, 5] that if G is a nilpotent group then the map α is always onto, due to the fact that any endomorphism of G inducing an automorphism of G/V, itself is an automorphism. We shall call this property (A); it will be a subject of our interest since it implies that α is onto.

For every two verbal subgroups U and V we define a subgroup $U^*(V)$, such that $[U, V] \subseteq U^*(V) \subseteq U \cap V$. For $U=V$ we get V^* and define inductively V^{n*} and V_{n*} so that the series

$$V \supseteq V^* \supseteq V^{2*} \supseteq V^{3*} \supseteq \cdots \tag{1}$$

and

$$V \supseteq V_* \supseteq V_{2*} \supseteq V_{3*} \supseteq \cdots \tag{2}$$

give some information about the map $\alpha:\operatorname{Aut} G \to \operatorname{Aut} G/V$. If either of the series ends in a finite number of steps at e, then the property (A) holds. If (1) ends at e, then V and $\operatorname{Ker}\alpha$ are soluble, and if (2) ends at e, then V and $\operatorname{Ker}\alpha$ are nilpotent. With the use of (1) we can show that if G is residually nilpotent and G/V is a Hopf group then G also is a Hopf group. In the case when $V=G'$, (2) coincides with $\gamma_2 \supseteq \gamma_3 \supseteq \gamma_4 \supseteq \cdots$, where we denote $\gamma_2=G'$, $\gamma_n=[\gamma_{n-1}, G]$, and also $\Gamma^1=G'$, $\Gamma^n=[\Gamma^{n-1}, \Gamma^{n-1}]$.

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2. Property (A)

Definition. The map $\alpha:\operatorname{Aut} G\to\operatorname{Aut} G/V$ has property (A) if every endomorphism of G which induces an automorphism of G/V, itself is an automorphism.

It is obvious that if α satisfies the property (A) then α is onto and $\operatorname{Ker}\alpha=\mathbf{1}+(V)\subseteq\operatorname{Aut} G$.

Lemma 1. *The property (A) holds if and only if $\mathbf{1}+(V)\subseteq\operatorname{Aut} G$.*

Proof. We need to show only that if $\mathbf{1}+(V)\subseteq\operatorname{Aut} G$ then every endomorphism $\mathbf{u}\in\operatorname{End} G$ which induces an automorphism $\bar{\mathbf{u}}\in\operatorname{Aut} G/V$ belongs to $\operatorname{Aut} G$. Let $\mathbf{u}_1\in\operatorname{End} G$ induce $\bar{\mathbf{u}}^{-1}\in\operatorname{Aut} G/V$, then $\mathbf{u}\mathbf{u}_1=\mathbf{1}+\mathbf{v}\in\operatorname{Aut} G$ and $\mathbf{u}\mathbf{u}_1(\mathbf{1}+\mathbf{v})^{-1}=\mathbf{1}$. Similarly, $\mathbf{u}$ is invertible from the left side and hence $\mathbf{u}\in\operatorname{Aut} G$.

Corollary. *Let $U\subseteq V$ and $\operatorname{Aut} G\xrightarrow{\alpha}\operatorname{Aut} G/U\xrightarrow{\beta}\operatorname{Aut} G/V$, then:*

1. *If (A) holds for α and β then it holds for $\alpha\beta$.*
2. *If (A) holds for $\alpha\beta$ then it holds for α.*

Proof. (1) If $\bar{\bar{\mathbf{u}}}\in\operatorname{Aut} G/V$ is induced by $\mathbf{u}\in\operatorname{End} G$, then $\mathbf{u}\xrightarrow{\alpha}\bar{\mathbf{u}}\xrightarrow{\beta}\bar{\bar{\mathbf{u}}}$. Because of (A) for β we get $^{-}\in\operatorname{Aut} G/U$ and because of (A) for α $\mathbf{u}\in\operatorname{Aut} G$. The statement (2) follows from Lemma 1, since $\mathbf{1}+(U)\subseteq\mathbf{1}+(V)\subseteq\operatorname{Aut} G$.

Lemma 2. *If $\alpha:\operatorname{Aut} G\to\operatorname{Aut} G/V$ satisfies (A) and G/V is a Hopf group, then G is also a Hopf group.*

Proof. Let $\operatorname{Sur} G$ be the semigroup of surjective endomorphisms of G. Then $\alpha:\operatorname{Sur} G\to\operatorname{Sur} G/V\subseteq\operatorname{Aut} G/V$, and because of (A), $\operatorname{Sur} G\subseteq\operatorname{Aut} G$.

3. Star subgroups

Definition. For every pair of verbal subgroups U and V in G we define $U^*(V)$ as the verbal subgroup generated by all the elements $u^*(\mathbf{v})=u^{-1}(x_1,x_2,\ldots,x_n)\;u(x_1v_1,x_2v_2,\ldots,x_nv_n)$, or briefly $u^*(\mathbf{v})=u^{-1}u(\mathbf{1}+\mathbf{v})$ for all $u\in U$, $\mathbf{v}\in(V)$.

It follows from the definition that if $U\subseteq W$, then $U^*(V)\subseteq W^*(V)$ and $V^*(U)\subseteq V^*(W)$.

Lemma 3. $[U,V]\subseteq U^*(V)\subseteq U\cap V$.

Proof. We take any $[u,v]$. If G has infinite rank then there exists x_i (say x_1) which does not occur in u. Let $\mathbf{w}=(x_1^{-1}vx_1,e,e,\ldots)\in(V)$, then $U^*(V)\ni[u,x_1]^*(\mathbf{w})=[x_1,u]\;[u,vx_1]=x_1^{-1}[u,v]x_1$ which gives $[U,V]\subseteq U^*(V)$. For G finitely generated the same follows because of [7, 13.42]. The second inclusion holds since U and V are verbal subgroups with the use of [7, 22.34].

Lemma 4. $[U,W]^*(V)\subseteq[U^*(V),W]\,[U,W^*(V)]$.

Proof. It is enough to check $[u,w]^*(\mathbf{v})=[w,u]u^{-1}(\mathbf{1}+\mathbf{v})\ uu^{-1}w^{-1}(\mathbf{1}+\mathbf{v})\ ww^{-1}uu^{-1}u(\mathbf{1}+\mathbf{v})ww^{-1}w(\mathbf{1}+\mathbf{v}) = [w,u]u^{*-1}u^{-1}w^{*-1}w^{-1}uu^*ww^* \equiv [w,u]u^{*-1}w^{*-1}[u,w]u^*w^* \equiv e$ modulo $[U^*(V),W][U,W^*(V)]$.

Lemma 5. *If $U=U_1U_2$, then $U^*(V)=U_1^*(V)U_2^*(V)$.*

Proof. Let $u=u_1u_2$, then $u^*(\mathbf{v})=u^{-1}\ u(\mathbf{1}+\mathbf{v})=u_2^{-1}u_1^{-1}u_1(\mathbf{1}+\mathbf{v})u_2(\mathbf{1}+\mathbf{v})=u_2^{-1}u_1^*(\mathbf{v})u_2\, u_2^*(\mathbf{v})\in U_1^*(V)U_2^*(V)$, which proves the statement.

Corollary. *If (u_i) is a set of generators for U, then $U^*(V)$ is generated as a verbal subgroup by the elements $u_i^*(\mathbf{v})$, $\mathbf{v}\in(V)$.*

As an example of star subgroup we compute it for the members of the lower central series.

Lemma 6. $\gamma_j^*(\gamma_k)=\gamma_{j+k-1}$.

Proof. To show the inclusion "$\supseteq$" we take $u=[x_1,x_2,\ldots,x_j]$, $v=[x_{j+1},x_{j+2},\ldots,x_{j+k}]$ and $\mathbf{v}=(v,e,e,\ldots)$. Then $u^*(\mathbf{v})=[x_1,x_2,\ldots,x_j]^{-1}[x_1v_1,x_2,\ldots,x_j]$. By δ_1 we denote the endomorphism such that $x_1\delta_1=e$, $x_i\delta_1=x_i$, $i\neq 1$. Now $\gamma_j^*(\gamma_k)\in(u^*(\mathbf{v}))\delta_1=[v,x_2,x_3,\ldots,x_j]$ and hence $\gamma_j^*(\gamma_k)\supseteq\gamma_{j+k-1}$. To prove the opposite inclusion we need to show, because of the Corollary to Lemma 5, that $u^*(\mathbf{v})\in\gamma_{j+k-1}$ only for $u=[x_{i_1},x_{i_2},\ldots,x_{i_s}]$, for $s\geqq j$. The commutator $[x_{i_1}v_{i_1},x_{i_2}v_{i_2},\ldots,x_{i_s}v_{i_s}]$ is a product of left-normed commutators with the components equal to x_i or v_i, where only one of the commutators has all the components equal to x_i and coincides with u. So, $u^*(\mathbf{v})=[x_{i_1},\ldots,x_{i_s}]^{-1}[x_{i_1}v_{i_1},\ldots,x_{i_s}v_{i_s}]$ belongs to γ_{j+k-1}. If G is finitely generated, the result follows with the use of [7, 13.42].

We denote $V^*(V)$ by V^* or by V_*, then inductively $V^{n*}=(V^{n-1*})^*(V^{n-1*})$ and $V_{n*}=(V_{n-1*})^*(V)$. Now because of Lemma 3, we get by induction the following:

Corollary. $\Gamma^n(V)\subseteq V^{n*}$.

Proof. $\Gamma^1(V)=[V,V]\subseteq V^*$ and $\Gamma^n(V)=[\Gamma^{n-1},\Gamma^{n-1}]\subseteq[V^{n-1*},V^{n-1*}]\subseteq V^{n*}$.

From Lemma 6 follows by induction:

Corollary. $(\gamma_k)^{n*}=\gamma_s$ *for* $s=2^n(k-1)+1$, $(\gamma_k)_{n*}=\gamma_s$ *for* $s=n(k-1)+k$.

Theorem 1. *Endomorphisms of G inducing the identity map in G/V commute if and only if $V^*=e$. The property (A) follows.*

Proof. We note that the equality $(\mathbf{1}+\mathbf{u})(\mathbf{1}+\mathbf{v})=(\mathbf{1}+\mathbf{v})(\mathbf{1}+\mathbf{u})$ is equivalent to $\mathbf{1}+\mathbf{v}+\mathbf{u}+\mathbf{u}^*(\mathbf{v})=\mathbf{1}+\mathbf{u}+\mathbf{v}+\mathbf{v}^*(\mathbf{u})$, where $u^*(\mathbf{v})=-\mathbf{u}+\mathbf{u}(\mathbf{1}+\mathbf{v})$ has components equal to $u_i^*(\mathbf{v})$. We conclude now that $\mathbf{1}+(V)$ is abelian if and only if $\mathbf{u}^*(\mathbf{v})-\mathbf{v}^*(\mathbf{u})=-\mathbf{u}-\mathbf{v}+\mathbf{u}+\mathbf{v}$ for all $\mathbf{u}\in(U)$, $\mathbf{v}\in(V)$. While written in components it gives

$$u_i^*(\mathbf{v})v_i^{*-1}(\mathbf{u})=[u_i,v_i]. \tag{3}$$

Let now $\mathbf{1}+(V)$ be abelian. We take any $u \in V$ and $\mathbf{v} \in (V)$. If G is infinitely generated there exists x_i (say x_1) which does not occur in $u=u(x_2, x_3, \dots, x_n)$. We take $\mathbf{u}=(u, e, e, \dots)$, $\mathbf{v}=(e, v_2, v_3, \dots)$ and consider the equality (3) for $i=1$, which is $u^*(\mathbf{v})=e$. This implies $V^*=e$. Conversely, if $V^*=e$, then by Lemma 3, $[V, V] \subseteq V^*=e$ and both sides of (3) are trivially equal and hence $\mathbf{1}+(V)$ is abelian. If G is finitely generated we get the statement with the use of [7, 13.42]. The property (A) holds because of Lemma 1, since $\mathbf{1}-\mathbf{v}$ is inverse to $\mathbf{1}+\mathbf{v}$ modulo V^*.

Corollary. *For* α:Aut $G/V^* \to$ Aut G/V *the property* (A) *holds and* Ker α *is abelian.*

Theorem 2. *The condition* $V^{n*}=e$ *is sufficient for the property* (A) *to hold.*

Proof. In the sequence of maps

$$\operatorname{Aut} G \to \operatorname{Aut} G/V^{n-1*} \to \operatorname{Aut} G/V^{n-2*} \to \cdots \operatorname{Aut} G/V^* \to \operatorname{Aut} G/V$$

all the maps have, by the previous corollary, the property (A). The statement follows now, since (A) is transitive by the Corollary to Lemma 1.

Problem. Does the property (A) imply $V^{n*}=e$?

Corollary. *Let* G *be residually nilpotent and* G/V *be a Hopf group, then* G *is a Hopf group.*

Proof. It follows from Theorem 2 that the map $\operatorname{Aut} G/V^{k*} \to \operatorname{Aut} G/V$ has the property (A) and hence, by Lemma 2, every G/V^{k*} is a Hopf group. Let now $\mathbf{u} \in \operatorname{Sur} G$, then $\operatorname{Ker} \mathbf{u} \subseteq \bigcap_k V^{k*}=e$.

Theorem 3. *Let* α:Aut $G \to$ Aut G/V. *If* $V^{n*}=e$, *then* Ker α is *soluble of length* $\leqq n$ *and also* V *is soluble of length* $\leqq n$.

Proof. We consider the sequence of maps

$$\operatorname{Aut} G \to \operatorname{Aut} G/V^{n-1*} \to \cdots \operatorname{Aut} G/V^* \to \operatorname{Aut} G/V$$

and the series of kernels of the maps of Aut G onto Aut G/V^{k*} for $k \geqq 0$

$$\mathbf{1}+(V) \rhd \mathbf{1}+(V^*) \rhd \mathbf{1}+(V^{2*}) \rhd \cdots \rhd \mathbf{1}+(V^{n-1*}) \rhd \mathbf{1}.$$

Following S. Bachmuth [1], we denote by $A(G/V^{k*}, G/V^{k-1*})$ the kernel of the map Aut G/V^{k*} onto Aut G/V^{k-1*}. Since (A) holds for each of the maps above we get by the Isomorphism Theorems $(\mathbf{1}+(V^{k*}))/(\mathbf{1}+(V^{k-1*})) \subseteq A(G/V^{k*}, G/V^{k-1*})$ which is abelian by the Corollary to Theorem 1. This imples $[\mathbf{1}+(V^{k*}), \mathbf{1}+(V^{k*})] \subseteq \mathbf{1}+(V^{k-1*})$ and hence $\mathbf{1}+(V)$ is soluble of length $\leqq n$. Since $\Gamma^n(V) \subseteq V^{n*}=e$, V is also soluble of length $\leqq n$, which completes the proof.

Theorem 4. *Let α:Aut $G \to$ Aut G/V. If $V_{n*}=e$, then the property (A) holds and* Ker α *and V are nilpotent of class $\leqq n$.*

Proof. Since $V^{n*} \subseteq V_{n*}=e$, by Theorem 2 the property (A) holds. The proof of the nilpotency of Ker α and V can be found in numerous papers by means of a different terminology [**2, 3, 8, 9**]. In fact, because of (A) Ker $\alpha = \mathbf{1}+(V) \subseteq$ Aut G is a subgroup in the holomorph of G. It can be computed by the definition that $[G, \mathbf{1}+(V)]=V$ and $[V_{k*}, \mathbf{1}+(V)] \subseteq V_{k+1*}$. We shall show now that $\gamma_{n+1}(V) \subseteq V_{n*}$. By Lemma 3 $\gamma_2(V) \subseteq V_*$, then by induction $\gamma_{n+1}(V)=[\gamma_n, V]=[\gamma_n, [G, \mathbf{1}+(V)]] \subseteq [G, \gamma_n, \mathbf{1}+(V)][\gamma_n, \mathbf{1}+(V), G] \subseteq [V_{n-1*}, \mathbf{1}+(V)] \subseteq V_{n*}$. Thus we have proven the equality which gives the nilpotency of V when $V_{n*}=e$. Similarly, by induction we can prove that $[V_{k*}, \gamma_j(\mathbf{1}+(V))] \subseteq V_{k+j*}$ and then, again by induction, $[G, \gamma_{n+1}(\mathbf{1}+(V))] \subseteq V_{n*}$, which implies the nilpotency of $\mathbf{1}+(V)$ when $V_{n*}=e$. It should be noted here that if $V_{n*}=e$, then Ker $\alpha = \mathbf{1}+(V)$ is a stability group for the normal series $G \rhd V \rhd V_* \rhd V_{2*} \rhd \cdots \rhd V_{n-1*} \rhd e$ of length $n+1$, which explains the nilpotency of V and Ker α of class $\leqq n=(n+1)-1$.

5. Examples

Theorem 5. *For any n the map α:Aut $G/(\gamma_{n+1} \cap V) \to$ Aut G/V has the property (A) and* Ker α *is nilpotent of class $\leqq [n/(k-1)]$, if $V \subseteq \gamma_k$.*

Proof. Since $V \subseteq G'$, there exists c such that $V_{c*} \subseteq \gamma_{n+1}$, then by Theorem 4, the property (A) holds. Now, from $V_{c*} \subseteq (\gamma_k)_{c*} = \gamma_{c(k-1)+k} \subseteq \gamma_{n+1}$ we compute c.

Theorem 6. *The map α:Aut $G/[\gamma_{k+i}, \gamma_k] \to$ Aut $G/[\gamma_k, \gamma_k]$ has the property (A) for any i and the* Ker α *is abelian if and only if $i \leqq 2k-1$.*

Proof. By Lemmas 4 and 6 $[\gamma_k, \gamma_k]^* \subseteq [\gamma_k, \gamma_k]^*(\gamma_{2k}) \subseteq [\gamma_k^*(\gamma_{2k}), \gamma_k] \subseteq [\gamma_{3k-1}, \gamma_k]$ and the statement follows from the Corollary to Theorem 1 and Theorem 2.

Theorem 7. *The map α:Aut $G/\gamma_n(V) \to$ Aut $G/[V, V]$ has the property (A) and* Ker α *is nilpotent of class $\leqq n-2$.*

Proof. The statement follows from Theorem 4 if we prove that $[V, V]_{n-2*} \subseteq \gamma_n(V)$. We need first to show that $(\gamma_{n-1}(V))^*([V, V]) \subseteq \gamma_n(V)$. For $n=2$, $V^*([V, V]) \subseteq \gamma_2(V)$ because of Lemma 3. Now, by induction with the use of Lemma 4 we get

$$\begin{aligned}(\gamma_{n-1}(V))^*([V, V]) &= [\gamma_{n-2}(V), V]^*([V, V]) \\ &\subseteq [(\gamma_{n-2}(V))^*([V, V]), V][\gamma_{n-2}(V), V^*([V, V])] \\ &\subseteq [\gamma_{n-1}(V), V][\gamma_{n-2}(V), [V, V]] \subseteq \gamma_n(V).\end{aligned}$$

At last, again by induction $[V, V]_{n-2*} = ([V, V]_{n-3*})^*([V, V]) \subseteq (\gamma_{n-1}(V))^*([V, V]) \subseteq \gamma_n(V)$, which finishes the proof.

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