

On symmetric words in nilpotent groups

by

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Abstract. An r -ary word $w(x_1, \dots, x_r)$ is called *symmetric in a group G* if for any elements g_1, \dots, g_r in G , and any permutation σ of indexes: $w(g_1, \dots, g_r) = w(g_{\sigma(1)}, \dots, g_{\sigma(r)})$. Symmetric r -ary words build a group $S^{(r)}(G)$. We shall show that $S^{(r)}(G)$ and $S^{(r+1)}(G)$ are isomorphic for G nilpotent of the class n and $r \geq n-1$.

Introduction. Let X be the free group on the set of free generators $\{x_i, i \in N\}$, where N denotes the set of natural numbers, and let V be a verbal subgroup of X . A word w from X is called an *r -ary word* if the reduced form of w involves r generators, say $x_{i_1}, x_{i_2}, \dots, x_{i_r}$. We then write $w = w(x_{i_1}, \dots, x_{i_r})$. For any permutation σ of N we denote $w\sigma = w(x_{\sigma(i_1)}, \dots, x_{\sigma(i_r)})$, and say that the words w and $w\sigma$ are equivalent under σ .

DEFINITION 1. A word $s(x_{i_1}, \dots, x_{i_r})$ from X is called *symmetric modulo V* if for every permutation σ_0 of $\{i_1, \dots, i_r\}$, $s\sigma_0 \equiv s \pmod{V}$.

We can see that the symmetry of $s(x_{i_1}, \dots, x_{i_r})$ implies the symmetry of all equivalent words $s\sigma = s(x_{i_{\sigma(1)}}, \dots, x_{i_{\sigma(r)}})$. Indeed if σ_0 is a permutation of the set $\{i_1, \dots, i_r\}$, then $\sigma\sigma_0\sigma^{-1}$ is a permutation of $R = \{1, 2, \dots, r\}$ and because of the symmetry of s , $s(\sigma\sigma_0\sigma^{-1}) \equiv s \pmod{V}$. This implies $(s\sigma)\sigma_0 \equiv s\sigma \pmod{V}$, which proves the symmetry of $s\sigma$.

According to [3] the set of all cosets sV , where $s = s(x_{i_1}, \dots, x_{i_r})$ is an r -ary word symmetric modulo V , written on $x_i, i \in R = \{1, 2, \dots, r\}$ forms a group $S^{(r)}(V)$. We shall suppose V fixed and write $S^{(r)}$. It will be shown that if $r \neq q$, then $S^{(r)} \cap S^{(q)} = 1 \pmod{V}$. Let $sV \in S^{(r)}, tV \in S^{(q)}, q < r$, and for some $v \in V$ $s(x_{i_1}, \dots, x_{i_r}) = t(x_{j_1}, \dots, x_{j_q})v(x_{i_1}, \dots, x_{i_r})$. Then for any $y_i, i \in R$, because of the symmetry of s we have

$$\begin{aligned} s(x_{i_1}, \dots, x_{i_r}) &= t(x_{j_1}, \dots, x_{j_q})v(x_{i_1}, \dots, x_{i_r}) \equiv s(x_{i_1}, \dots, x_{r-1}, y_1) \\ &\equiv s(y_1, x_1, \dots, x_{r-1}) = t(y_1, x_1, \dots, x_{q-1})v(y_1, x_1, \dots, x_{r-1}) \\ &\equiv s(y_1, x_1, \dots, x_{r-2}, y_2) \equiv s(y_1, y_2, x_1, \dots, x_{r-2}) \\ &= t(y_1, y_2, x_1, \dots, x_{q-2})v(y_1, y_2, x_1, \dots, x_{r-2}) \equiv \dots \\ &\dots \equiv s(y_1, y_2, \dots, y_q) \pmod{V}. \end{aligned}$$

Put $y_i = 1, i \in R$, then $s(x_1, \dots, x_r) = V$ which was required.

Since V is a fully invariant subgroup of X , the mapping

$$\delta_{r+1}: s(x_1, \dots, x_r, x_{r+1}) \rightarrow s(x_1, \dots, x_r, 1)$$

determines a homomorphism $\delta_r^{r+1}: S^{(r+1)} \rightarrow S^{(r)}$. In [3] it is shown that the homomorphism δ_r^{r+1} is an isomorphism if $V = X^{(n)}$, and $r \geq n-1$. Here $X^{(1)} = X$, $X^{(k)} = [X^{(k-1)}, X]$. In [4] the same result is proved for $V \supseteq X^{(n)}$, $n \leq 4$, $r \geq n-1$, and the problem is formulated whether the mapping δ_r^{r+1} is an isomorphism for every n , $V \supseteq X^{(n)}$, $r \geq n-1$.

We give here an affirmative solution of the above problem.

Definitions and lemmas. In X we introduce an endomorphism δ_k ($k > 0$) given by $x_k \delta_k = 1$, $x_i \delta_k = x_i$ for $i \neq k$. Clearly

$$(1) \quad \delta_i \delta_j = \delta_j \delta_i, \quad \delta_i^2 = \delta_i.$$

Any permutation σ of the set N induces an automorphism of X such that

$$(2) \quad \delta_i \sigma = \sigma \delta_{\sigma(i)}, \quad \sigma \delta_i = \delta_{\sigma^{-1}(i)} \sigma.$$

We shall write $D_i = \text{Ker} \delta_i$ and $D_R = \bigcap_{i \in R} D_i$.

DEFINITION 2. A word $w(x_1, \dots, x_r)$ is called *neutral* if $w \delta_i = 1$, $i \in R = \{1, 2, \dots, r\}$.

The set of all r -ary neutral words on generators x_i , $i \in R$, obviously coincides with D_R .

LEMMA 1. If $w(x_1, \dots, x_r)$ is a neutral r -ary word then $w \in X^{(r)}$.

Proof. $w \in D_R \subseteq X^{(r)}$ follows from ([2], 33.38).

LEMMA 2. If $s(x_1, \dots, x_r)$ is a neutral r -ary word, symmetric modulo $X^{(r+1)}$ then $s \in X^{(r+1)}$.

Proof. Denote by S_r a group of permutations of R , and by A_r its alternating subgroup. We shall consider the word $s_0 = \prod_{\sigma \in S_r} c\sigma$, where $c = [x_1, x_2, x_3, \dots, x_r]$ is the left-normed commutator, and the product is taken for some fixed order of factors. Obviously s_0 is a neutral r -ary word, symmetric modulo $X^{(r)}$. If we denote by $\sigma_{(1,2)}$ the cycle (1, 2) then $s_0 = \prod_{\sigma \in A_r} (c\sigma_{(1,2)}) \sigma \text{ mod } X^{(r+1)}$. With the use of a commutator calculus modulo $X^{(r+1)}$ we have

$$\begin{aligned} c\sigma_{(1,2)} &= [x_1, x_2, x_3, \dots, x_r] [[x_1, x_2]^{-1}, x_3, \dots, x_r] \\ &= cc^{-1} \text{ mod } X^{(r+1)} \in X^{(r+1)}. \end{aligned}$$

This implies $s_0 \in X^{(r+1)}$, which will be used later.

Now since $s(x_1, \dots, x_r)$ is a neutral r -ary word, by Lemma 1 $s \in X^{(r)}$, and by ([2], 34.21) s is a product modulo $X^{(r+1)}$ of commutators $c\sigma$ for some $\sigma \in S_r$, say $s = \prod c\sigma \text{ mod } X^{(r+1)}$. Now, since s is symmetric modulo $X^{(r+1)}$, for

every $\sigma_0 \in S_r$, $\prod c\sigma \equiv \prod c(\sigma\sigma_0) \text{ mod } X^{(r+1)}$. By Hall's basis theorem ([1], 11.2.4) we conclude that s is a power of s_0 and hence $s \in X^{(r+1)}$, which was required.

Now let $w \neq 1$ be an element of X . We write

$$(3) \quad w(1-\delta_i) = w(w\delta_i)^{-1}.$$

Then by (1)

$$(4) \quad (1-\delta_i)\delta_j = \delta_j(1-\delta_i), \quad (1-\delta_i)\delta_i = 1.$$

DEFINITION 3. For a word $w(x_1, \dots, x_r)$ and $k \leq r$, we define the k -ary image of w as $w_k = w(x_1, \dots, x_k, 1, \dots, 1) = w \prod_{i=k+1}^r \delta_i$. The neutral part of w

we define as $\bar{w} = w \prod_{i=1}^r (1-\delta_i)$. The word \bar{w} is neutral by ([2], 33.42).

For a set $M = \{i_1, i_2, \dots, i_k\} \subseteq N$ we shall always suppose $i_1 < i_2 < \dots < i_k$. We denote now $K = \{1, 2, \dots, k\}$ and introduce a permutation σ_M of the set $M \cup K$ such that $\sigma_M(j) = i_j$, $j \leq k$. Then $\sigma_M: K \rightarrow M$. If $M \subseteq R = \{1, 2, \dots, r\}$, then σ_M can be considered as a permutation of R , since $M \cup K \subseteq R$. In case $k = r$, σ_M is obviously the identical permutation. If $s(x_1, \dots, x_k)$ is a k -ary word then $s\sigma_M$ is an equivalent word written on generators with indices form M .

DEFINITION 4. We shall define a special order (*) for subsets in N . There exists on-to-one correspondence between subsets $M = \{i_1, i_2, \dots, i_k\}$, $i_1 < i_2 < \dots < i_k$, of N and formal sequences $\langle i_1, i_2, \dots, i_k, \omega, \omega, \dots \rangle$. We assume $\omega > i$ for every $i \in N$. The lexicographical order for the sequences induces the order for subsets in N . We shall refer to it as to the order (*).

LEMMA 3. Every symmetric word $s(x_1, \dots, x_r)$ is a product (modulo V) of the neutral parts \bar{s}_k of its k -images, $k \leq r$, and the equivalent words. More precisely $s = \prod \bar{s}_k \sigma_M$ is a product of 2^r factors corresponding to the subsets $M \subseteq R$, for $k = |M|$, taken in order (*).

Proof. We introduce an algorithm that allows us to write any word $s(x_1, \dots, x_r)$ as a product $\prod u_M$ of 2^r factors corresponding to the subsets $M \subseteq R$ in order (*), where $u_M = s \prod_{i \in M} (1-\delta_i) \prod_{j \in R \setminus M} \delta_j$. According to (3) for any word s and any $i \in N$

$$(5) \quad s = s(1-\delta_i) s \delta_i.$$

So for $i = 1$, $s = s(1-\delta_1) s \delta_1$. Now apply (5) for $i = 2$ to each factor separately. We get

$$s = s(1-\delta_1)(1-\delta_2) s(1-\delta_1) \delta_2 s \delta_1 (1-\delta_2) s \delta_1 \delta_2.$$

Apply (5) for $i = 3$ to each factor separately and so on. The result will be achieved in r steps with the use of (4). See ([2], 33.44).

We can see that for the case $r = 2$ the factors correspond to the subsets

$$\{1, 2\} < \{1\} < \{2\} < \{\emptyset\}.$$

Suppose that for the $(r-1)$ -st step the factors correspond to the subsets

$$M_1 < M_2 < \dots < M_{2^{r-1}} < \{\emptyset\},$$

then by applying (5) for δ_r we get a product of 2^r factors corresponding to the sequence of subsets

$$\{M_1, r\} < M_1 < \{M_2, r\} < M_2 < \dots < \{M_{2^{r-1}}, r\} < M_{2^{r-1}} < \{r\} < \{\emptyset\},$$

which was required.

To prove the lemma we shall show that if s is a symmetric word, then $u_M = \bar{s}_k \sigma_M$. Indeed, since $\sigma_M: R \rightarrow R$, $s \equiv s \sigma_M \pmod V$. Then by using (2) and (4)

$$\begin{aligned} u_M &= s \sigma_M \prod_{i \in M} (1 - \delta_i) \prod_{j \in R \setminus M} \delta_j = s \prod_{i \in M \sigma_M^{-1}} (1 - \delta_i) \prod_{j \in R \setminus M \sigma_M^{-1}} \delta_j \sigma_M \\ &= s \prod_{i \in K} (1 - \delta_i) \prod_{j \in R \setminus K} \delta_j \sigma_M = \left(s \prod_{i=k+1}^r \delta_i \right) \prod_{i=1}^k (1 - \delta_i) \sigma_M = \bar{s}_k \sigma_M \end{aligned}$$

as required.

For $M = \{i_1, i_2, \dots, i_k\}$ and $K = \{1, 2, \dots, k\}$ we have defined $\sigma_M: K \rightarrow M$. If take $M_0 = \{1, 2, \dots, i_0 - 1, i_0 + 1, \dots, r + 1\}$, and $R = \{1, 2, \dots, r\}$ then $\sigma_{M_0}: R \rightarrow M_0$ is a cycle $\sigma_{M_0} = (i_0, i_0 + 1, \dots, r, r + 1)$ which is a permutation of $J = \{1, 2, \dots, r + 1\}$. Here in the case $i_0 = r + 1$ σ_{M_0} is the identical permutation.

LEMMA 4. For a symmetric word $w(x_1, \dots, x_{r+1})$ denote $w \delta_{r+1} = s$, then

$$(6) \quad w \delta_{i_0} \equiv s \sigma_{M_0} \pmod V \quad \text{for } i_0 \leq r + 1.$$

$$(7) \quad \bar{w}_k = \bar{s}_k \quad \text{for } k \leq r.$$

For any neutral word $w(x_1, x_2, \dots, x_k)$

$$(8) \quad w \sigma_M \delta_{i_0} = \begin{cases} w \sigma_M & \text{if } i_0 \notin M, \\ 1 & \text{if } i_0 \in M. \end{cases}$$

Proof. Because of the symmetry of w and (2) $w \delta_{i_0} = w \sigma_{M_0} \delta_{i_0} = w \delta_{r+1} \sigma_{M_0} = s \sigma_{M_0}$ which gives (6). By Definition 3 and (4)

$$\bar{w}_k = w \prod_{i=k+1}^{r+1} \delta_i \prod_{j=1}^k (1 - \delta_j) = (w \delta_{r+1}) \prod_{i=k+1}^r \delta_i \prod_{j=1}^k (1 - \delta_j) = \bar{s}_k,$$

gives (7). Now by (2) $\sigma_M \delta_{i_0} = \delta_{\sigma_M^{-1}(i_0)} \sigma_M$. The index $\sigma_M^{-1}(i_0)$ belongs to K if and only if $i_0 \in M$. Since w is neutral, we have (8). The lemma is proved.

Theorems on the homomorphism $\mathcal{C}_r^{r+1}: S^{(r+1)} \rightarrow S^{(r)}$.

THEOREM 1. If $V \cong X^{(n)}$ then \mathcal{C}_r^{r+1} is an epimorphism for $r \geq n - 1$.

Proof. A coset $sV \in S^{(r)}$ has a conainment under \mathcal{C}_r^{r+1} if and only if there exists an $(r+1)$ -ary symmetric word w such that $wV \in S^{(r+1)}$ and $w \delta_{r+1} = s$. By Lemma 3, $w = \prod \bar{w}_k \sigma_M$, where $k = |M|$ and M runs over all non-empty subsets of the set $J = \{1, 2, \dots, r + 1\}$ in order (*). In this order the first factor of the product corresponds to $M = J$ and coincides with the neutral part of w , namely $\bar{w} = w \prod_{i \in J} (1 - \delta_i)$. Then $w = \bar{w} \prod \bar{w}_k \sigma_M$ where $M \subset J$, i.e. $k < r + 1$. We shall denote $w_0 = \prod \bar{w}_k \sigma_M$, $k < r + 1$. Since by Lemma 1 $\bar{w} \in X^{(r+1)} \subseteq V$, we have $w \equiv w_0 \pmod V$ and w_0 is completely defined by the word s because of (7).

Let $s(x_1, \dots, x_r)$ be a symmetric word. We construct $w_0 = \prod \bar{s}_k \sigma_M$ where M runs over the proper subsets of $J = \{1, 2, \dots, r + 1\}$ in order (*). We shall show that $w_0 V$ is a conainment of sV under \mathcal{C}_r^{r+1} . We shall first check the equality

$$(9) \quad w_0 \delta_{i_0} = s \sigma_{M_0} \pmod V \quad \text{for } i_0 \leq r + 1.$$

By (8) we have $w_0 \delta_{i_0} = \prod_{M \subset J} \bar{s}_k \sigma_M \delta_{i_0} = \prod_{M \in M_0} \bar{s}_k \sigma_M$ with factors in order (*).

We now consider $s \sigma_{M_0}$. Notice that $\sigma_{M_0}: R \rightarrow M_0$ gives a one-to-one correspondence for subsets of M_0 and R , preserving the order (*). So if M runs over the subsets of M_0 , then $M' = M \sigma_{M_0}^{-1}$ runs over the subsets of R and $s = \prod_{M' \in R} \bar{s}_k \sigma_{M'}$. Since $\sigma_{M'}: K \rightarrow M' \subseteq R$ and $\sigma_{M_0}: R \rightarrow M_0$ we have $K \xrightarrow{\sigma_{M'}} M \sigma_{M_0}^{-1} \xrightarrow{\sigma_{M_0}} M \sigma_{M_0}^{-1} \sigma_{M_0} = M$, which means that $\sigma_{M'} \sigma_{M_0}$ coincides with σ_M on K and hence $s \sigma_{M_0} = \prod_{M' \in R} \bar{s}_k \sigma_{M'} \sigma_{M_0} = \prod_{M \in M_0} \bar{s}_k \sigma_M$. So (9) is proved. We notice that $\sigma_{M_0} = (i_0, i_0 + 1, \dots, r, r + 1) = (i_0, i_0 + 1, \dots, r)(i_0, r + 1) = \sigma \sigma_{(i_0, r+1)}$. The cycle σ is a permutation of R and because of the symmetry of s equality (9) can be written as

$$(10) \quad w_0 \delta_k = s \sigma_{(k, r+1)}, \quad k \leq r + 1.$$

We have shown that w_0 is an $(r+1)$ -ary word and by (10) $w_0 \delta_{r+1} = s$. Now we have to check the symmetry of w_0 modulo V . Since every permutation is a product of cycles it is enough to show $w_0 \sigma_{(i,j)} \equiv w_0 \pmod V$, for $i < j \leq r + 1$. We denote $v = w_0 \sigma_{(i,j)} w_0^{-1}$ and show first that for every $k \leq r + 1$ $v \delta_k \in V$. For this purpose we shall consider seven cases:

1. $i = k \neq j$; a. $j = r + 1$, b. $j \neq r + 1$.
2. $i \neq k = j$; a. $j = r + 1$, b. $j \neq r + 1$.
3. $i \neq k \neq j$; a. $k = r + 1$, b. $j = r + 1$, c. $j \neq r + 1 \neq k$.

By (2) and (10) we have

$$1. v\delta_k = w_0 \sigma_{(k,j)} \delta_k w_0^{-1} \delta_k = w_0 \delta_j \sigma_{(k,j)} w_0^{-1} \delta_k = s\sigma_{(j,r+1)} \sigma_{(k,j)} s^{-1} \sigma_{(k,r+1)}.$$

In case 1a. $j = r+1$, $v\delta_k = s\sigma_{(k,r+1)} s^{-1} \sigma_{(k,r+1)} = 1$.

In case 1b. $j \neq r+1$,

$$v\delta_k = s\sigma_{(j,r+1)} \sigma_{(k,j)} s^{-1} \sigma_{(k,r+1)} \\ = s\sigma_{(k,j)} \sigma_{(k,r+1)} s^{-1} \sigma_{(k,r+1)} = (s\sigma_{(k,j)} s^{-1}) \sigma_{(k,r+1)} \in V,$$

because of the symmetry of s , since $k, j < r+1$.

$$2. v\delta_k = w_0 \sigma_{(i,k)} \delta_k w_0^{-1} \delta_k = w_0 \delta_i \sigma_{(i,k)} w_0^{-1} \delta_k = s\sigma_{(i,r+1)} \sigma_{(i,k)} s^{-1} \sigma_{(k,r+1)}.$$

In case 2a. $k = j = r+1$, $v\delta_k = ss^{-1} = 1$.

In case 2b. $k = j \neq r+1$,

$$v\delta_k = s\sigma_{(i,r+1)} \sigma_{(i,k)} s^{-1} \sigma_{(k,r+1)} \\ = s\sigma_{(i,k)} \sigma_{(k,r+1)} s^{-1} \sigma_{(k,r+1)} = (s\sigma_{(i,k)} s^{-1}) \sigma_{(k,r+1)} \in V.$$

$$3. v\delta_k = w_0 \sigma_{(i,j)} \delta_k w_0^{-1} \delta_k = w_0 \delta_k \sigma_{(i,j)} w_0^{-1} \delta_k = s\sigma_{(k,r+1)} \sigma_{(i,j)} s^{-1} \sigma_{(k,r+1)}.$$

In case 3a. $k = r+1$, $v\delta_k = s\sigma_{(i,j)} s^{-1} \in V$.

In case 3b. $j = r+1$,

$$v\delta_k = s\sigma_{(k,r+1)} \sigma_{(i,r+1)} s^{-1} \sigma_{(k,r+1)} \\ = s\sigma_{(i,k)} \sigma_{(k,r+1)} s^{-1} \sigma_{(k,r+1)} = (s\sigma_{(i,k)} s^{-1}) \sigma_{(k,r+1)} \in V.$$

In case 3c. $j \neq r+1 \neq k$, $v\delta_k = (s\sigma_{(i,j)} s^{-1}) \sigma_{(k,r+1)} \in V$. We have shown that $v\delta_k \in V$, $k \leq r+1$. By Definition 2 this implies the neutrality of v modulo V and by Lemma 1, $v \in X^{(r+1)} V \subseteq V$, since $X^{(r+1)} \subseteq X^{(n)} \subseteq V$. Now $v \in V$ implies the required symmetry of w_0 modulo V . So $w_0 V$ is a contraimage of sV under $\tilde{\sigma}_r^{+1}$ and the proof is complete.

THEOREM 2. *If $V \supseteq X^{(n)}$, then $\tilde{\sigma}_r^{+1}$ is an isomorphism for $r \geq n-1$.*

PROOF. If $wV \in \text{Ker } \tilde{\sigma}_r^{+1}$ then $w\delta_{r+1} = 1 \pmod V$ and by (6) $w\delta_i = 1 \pmod V$ for $i \leq r+1$. Hence w is a neutral $(r+1)$ -ary word modulo V and, by Lemma 1, $w \in X^{(r+1)} V \subseteq V$. This means that $\tilde{\sigma}_r^{+1}$ is a monomorphism, which completes the proof because of Theorem 1.

THEOREM 3. *If $V = X^{(n)}$, then $\tilde{\sigma}_r^{+1}$ is a monomorphism for $r \geq n-2$, and is not a monomorphism for $r < n-2$.*

PROOF. For $r \geq n-1$ the statement follows from Theorem 2. If $r = n-2$ and $wV \in \text{Ker } \tilde{\sigma}_r^{+1}$, then w is an $(r+1)$ -ary word neutral modulo $V = X^{(r+2)}$, and by Lemma 2 $w \in X^{(r+2)} = V$. This means that $\text{Ker } \tilde{\sigma}_r^{+1}$ is trivial.

Let $r < n-2$. Denote by $d = [x_2, x_1, x_1, \dots, x_1, x_3, x_4, \dots, x_{r+1}]$ a left-normed commutator of the weight k , with x_1 repeated at least twice. If x_1 is repeated twice then $k = r+2 < n$. We shall also suppose $\frac{1}{2}n \leq k < n$ repeating x_1 if necessary. The word $d_0 = \prod_{\sigma \in S_{r+1}} d\sigma$ is obviously an $(r+1)$ -ary neutral word symmetric modulo $X^{(2k)} \subseteq X^{(n)} = V$, hence $d_0 V \in \text{Ker } \tilde{\sigma}_r^{+1}$. We

have to check that $d_0 \notin V$. By the use of the commutator calculus (see [1]) modulo $X^{(k+3)}$ and the identity which follows from ([1], 10.2.1.4): $[[x, y], z] = [x, [y, z]] [[x, z], y] \pmod{X^{(4)}}$, we can write d_0 as a product of basic commutators (see [1]) modulo $X^{(k+3)}$. The process is based on the typical step: $t = [x_i, x_j, \dots, x_k, x_l]$ is a basic commutator, $i > j < \dots < k < l$, and $l > m$, then $[t, x_m] = [[x_i, x_j, \dots, x_k], [x_l, x_m]] [x_i, x_j, \dots, x_k, x_m, x_l]$. It can be shown by induction that $d\sigma$ written as a product of basic commutators involves d if and only if $\sigma(1) = 1$, $\sigma(2) = 2$. Then d_0 written as a product of basic commutators modulo $X^{(r+3)}$ contains d to the power $(r-1)!$, hence $d_0 \notin X^{(r+3)}$ and $d_0 \notin V$, which completes the proof.

References

- [1] M. Hall, *The theory of groups* (Russian translation), Moskwa 1962.
- [2] H. Neumann, *Varieties of groups* (Russian translation), Moskwa 1969.
- [3] E. Płonka, *On symmetric words in free nilpotent groups*, Bull. Acad. Polon. Sci. 18 (8) (1970), pp. 427-429.
- [4] — *Symmetric words in nilpotent groups of class ≤ 3* , Fund. Math. 97 (1977), pp. 95-103.

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