Commutators and powers of infinite unitriangular matrices

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Abstract

In the paper we consider some commutator-type and power-type matrix equations in the group $\mathrm{UT}(\infty, K)$ of infinite dimensional unitriangular matrices over a field K. We introduce a notion of a *power outer commutator* $\omega_k^{m_1,\dots,m_k}(x_1, \dots, x_k)$ and a *power Engel commutator* $e_k^{l,m_1,\dots,m_k}(x, y)$ as outer (respectively Engel) commutators modified by allowing powers of letters instead of letters alone.

For a given infinite unitriangular matrix A we discuss the matrix equations $x^k = A$, $\omega_k^{m_1,\dots,m_k}(x_1, \dots, x_k) = A$ and $e_k^{l,m_1,\dots,m_k}(x, y) = A$ in variables $x, x_1, \dots, x_k, y$. As a main result, we provide the necessary and sufficient conditions for solvability of these equations.

Keywords: infinite unitriangular matrices, verbal subgroup, commutator width, Engel width, power width
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1. Introduction

Let K be an arbitrary field. Throughout the paper we assume $charK \neq 2$. By $\mathrm{UT}(n, K)$ we denote the group of upper unitriangular matrices of size $n \times n$ over the field K, i.e. the upper triangular matrices with all diagonal entries equal to 1. We are interested in the infinite dimensional generalization $\mathrm{UT}(\infty, K)$ of the group $\mathrm{UT}(n, K)$, which consists of all upper unitriangular matrices with entries indexed by elements of the set $\mathbb{N} \times \mathbb{N}$.

When considering the infinite dimensional generalizations of matrix groups, it is usually necessary to impose additional conditions on a given set of infinite matrices, which yield its closure under the group operations (multiplication and taking an inverse). One possible generalization of linear groups is the group $GL_{cf}(\infty, K)$ of column finite infinite dimensional matrices over the field K, consisting of matrices in which every column contains only finitely many nonzero entries. Clearly, the set of all column finite matrices is closed under matrix multiplication, however the closure under taking inverses must be additionally imposed to get a group structure [8]. So, $GL_{cf}(\infty, K)$ consists of exactly those infinite column finite matrices, whose inverses are column finite. The subgroups of this group were investigated e.g. in [7] and [9]. Dually, one may consider

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the group $GL_{rf}(\infty, K)$, consisting of row finite infinite dimensional matrices over the field K with row finite inverses. The intersection $GL_{rcf}(\infty, K) = GL_{cf}(\infty, K) \cap GL_{rf}(\infty, K)$ is a group of row-column-finite infinite dimensional matrices and is discussed in [20],[21]. A subgroup of $GL_{rcf}(\infty, K)$ of all infinite matrices, which differ form the infinite identity matrix on finitely many entries, is called the *stable* linear group and denoted by $GL_f(\infty, K)$. This group is an object of investigations in many aspects, such as the characterization of its Sylow p-subgroups (see [10–12]) or the automorphism group of $GL_f(\infty, K)$ [1].

It is clear that every matrix from $\mathrm{UT}(\infty, K)$ is column finite and its inverse is an infinite unitriangular matrix. Hence $\mathrm{UT}(\infty, K)$ is a subgroup of $GL_{cf}(\infty, K)$. The intersection of $\mathrm{UT}(\infty, K)$ with the stable linear group is a subgroup called the *group of finitary unitriangular matrices* over the field K and denoted $\mathrm{UT}_f(\infty, K)$.

In the presented paper we discuss the solvability of matrix equations of a specific type in the group $\mathrm{UT}(\infty, K)$. The problem of existence of solutions to various types of matrix equations has been investigated by many. For example, equations of the form $f(x) = A$ in $GL(n, \mathbb{C})$, with given matrix A and a complex holomorphic function f, are investigated in [5]. The existence of solutions x of the power equations of the type $x^m = A$ for the given matrix A, i.e. the existence of roots of matrices of degree m, was considered in [3], [15] and [16]. In [15] and [16] the authors discuss the solvability of power equations in the groups $GL(n, K)$ in terms of the characteristic polynomial of matrix A. In [3] the necessary and sufficient conditions for solvability of power and commutator equations in groups $\mathrm{UT}(n, K)$ are provided. It is shown that the solvability of power equations in the discussed groups depends on the field characteristic. The problems of this kind are also investigated in some groups of infinite dimensional matrices. For instance, the existence of matrix inverses in the group $GL_{rcf}(\infty, K)$ is considered in [20]. Some specific commutator equations in groups of infinite dimensional triangular matrices over a field were discussed in [4], [7] and [18]. Equations of this type involve several variables and provide useful information on the structure of the considered group. In this article we improve the subject in two directions. We investigate the solvability of generalized matrix equations which involve both powers and commutators. The problem is considered in the group of infinite unitriangular matrices $\mathrm{UT}(\infty, K)$, without assuming any additional finiteness conditions.

To state our results we introduce the necessary terminology and notation. In groups $\mathrm{UT}(n, K)$, $\mathrm{UT}(\infty, K)$ and $\mathrm{UT}_f(\infty, K)$ we distinguish the respective subgroups $\mathrm{UT}(n, m, K)$, $\mathrm{UT}(\infty, m, K)$ and $\mathrm{UT}_f(\infty, m, K)$, $m \geq 0$, consisting of exactly those matrices, whose all entries on the first m superdiagonals are zeros. It is known (see e.g. [13], Chapter 6, p. 145), that the series of subgroups:

$$\mathrm{UT}(n, K) = \mathrm{UT}(n, 0, K) > \mathrm{UT}(n, 1, K) > ... > \mathrm{UT}(n, n-1, K) = \{e_n\}$$

is the *lower central series* in $\mathrm{UT}(n, K)$. In the groups of infinite matrices the series

$$\mathrm{UT}(\infty, K) = \mathrm{UT}(\infty, 0, K) > \mathrm{UT}(\infty, 1, K) > ...$$

is obviously infinite, however an analogous statement holds in $\mathrm{UT}_f(\infty, K)$ and $\mathrm{UT}(\infty, K)$ (see [2] and [18]). The terms of the lower central series of a group

G are commonly denoted by $\gamma_m(G)$, $m \geq 1$ (see [17], p. 121). It is known that $\gamma_m(G)$ is generated by the so-called basic commutators c_m. A *commutator* of elements $g, h \in G$ is the element $[g, h] := g^{-1}h^{-1}gh$. Then the *basic commutator* $c_m(g_1, ..., g_m)$ is defined recursively as $c_1(g_1) := g_1$, $c_{i+1} := [c_i(g_1, g_2, ..., g_i), g_{i+1}]$, $g_i \in G$, $i \geq 1$. For the groups of unitriangular matrices we have ([2],[18]):

$$\gamma_m(\mathrm{UT}(n, K)) = \mathrm{UT}(n, m-1, K), \qquad \gamma_m(\mathrm{UT}(\infty, K)) = \mathrm{UT}(\infty, m-1, K).$$

The subgroups $\gamma_m(G)$ are examples of the so-called verbal subgroups of G. Given a group G and a set of words $W \subset \mathcal{F}(X)$ over a countable alphabet X (here $\mathcal{F}(X)$ denotes the free group generated freely by X), we define the *verbal subgroup* $W(G)$ of group G as the subgroup generated by all values of words from W in G. A *value of a word* $w \in W$ in group G is obtained by substituting all letters in the word w by elements from G and calculating the resulting product. We will use notation $w(G)$, if the verbal subgroup is generated by a single word $w \in \mathcal{F}(X)$, e.g. $\gamma_m(G) = c_m(G)$. We note that the set G^W of values of words from W in G is not necessarily equal to $W(G)$. If the equality $W(G) = G^W$ holds, then every element of the verbal subgroup $W(G)$ is a value from G^W; if it is not the case, then every element of $W(G)$ is a product of a finite number of values form G^W. The *width* $wid_W(G)$ of verbal subgroup $W(G)$ is defined as the smallest number k, such that every element of $W(G)$ can be written as a product of at most k values from G^W. It may happen that such a number does not exist, in that case the width is considered to be infinite.

The verbal subgroups and their width in groups of matrices were investigated in [3, 4, 19]. In [2] it was proved, that in the groups of finite and finitary unitriangular matrices, all verbal subgroups coincide with terms of the lower central series, i.e. with one of subgroups $\mathrm{UT}(n, m, K)$ (or $\mathrm{UT}_f(\infty, m, K)$, respectively). Moreover, in [3] and [4] the coincidence of $w(\mathrm{UT}(n, K))$ (resp. $w(\mathrm{UT}_f(\infty, K))$) and $\mathrm{UT}(n, m, K)$ (resp. $\mathrm{UT}_f(\infty, m, K)$) and the respective verbal width was determined for specific kinds of words w:

- the outer-commutators, defined recursively:

  $\omega_1(x_1) = x_1$ - the outer commutator of weight 1,

  $\omega_{i+j}(x_1, x_2, ..., x_{i+j}) = [\omega_i(x_1, x_2, ..., x_i), \omega_j(x_{i+1}, x_{i+2}, ..., x_{i+j}]$ - the outer commutator of weight $i + j$;

- the Engel commutators:

$$e_m(x, y) = [x, \underbrace{y, y, ..., y}_{m}] = [e_{m-1}(x, y), y], \quad m \geq 1;$$

- the powers $x^m = \underbrace{x \cdot x \cdot ... \cdot x}_{m}$.

The results in [3] provide examples of verbal subgroups in $\mathrm{UT}(n, K)$ generated by power words with verbal width greater than 1. In other words, there exist matrices $A \in \mathrm{UT}(n, K)$, where $char K = p \neq 2$, such that the power equations

$x^{p\cdot m} = A$, $m \in \mathbb{N}$, are not solvable in $\mathrm{UT}(n, K)$. An interesting question arises, whether the verbal subgroups defined by other words which contain powers would provide non solvable equations in groups of unitriangular matrices.

We investigate the commutators of powers, defined as:

- power outer commutators of weight k:

$$\omega_k^{m_1,...,m_k}(x_1, ..., x_k) := \omega_k(x_1^{m_1}, ..., x_k^{m_k}), \quad k \geq 1, \quad m_i \in \mathbb{Z} \setminus \{0\},$$

- power Engel commutators:

$$e_k^{l,m_1,...,m_k}(x, y) := [x^l, \underbrace{y^{m_1}, ..., y^{m_k}}_{k}], \quad k \geq 1 \quad l, m_i \in \mathbb{Z} \setminus \{0\}.$$

As a main result of the present paper we show, that in the groups of unitriangular matrices every matrix from the verbal subgroup generated by power outer commutators and power Engel commutators is the value of the respective generating word. We also determine the coincidence of the particular verbal subgroup with a given term of the lower central series of the group. This way we provide a criterion for the solvability of power outer/Engel commutator equations. The results are stated in the following:

THEOREM *Let G be either $\mathrm{UT}(n, K)$ or $\mathrm{UT}_f(\infty, K)$ or $\mathrm{UT}(\infty, K)$.*

1. *If K is a field of characteristic 0, then*

   (a) *for every power outer commutator $\omega_k^{m_1,...,m_k}$ the verbal subgroup $\omega_k^{m_1,...,m_k}(G)$ coincides with $\gamma_k(G)$ and has width equal to 1;*

   (b) *for every power Engel commutator $e_k^{l,m_1,...,m_k}$ the verbal subgroup $e_k^{l,m_1,...,m_k}(G)$ coincides with $\gamma_{k+1}(G)$ and has width equal to 1.*

2. *If K is a field of positive characteristic $p \neq 2$, then*

   (a) *for every power outer commutator $\omega_k^{m_1,...,m_k}$, such that*

   $$m_i = p^{\alpha_i} \cdot r_i, \quad p \nmid r_i, \quad i = 1, 2, ..., k,$$

   the verbal subgroup $\omega_k^{m_1,...,m_k}(G)$ coincides with $\gamma_s(G)$, where $s = \sum_{i=1}^{k} p^{\alpha_i}$, and $\mathrm{wid}_{\omega_k^{m_1,...,m_k}}(G) = 1$;

   (b) *for every power Engel commutator $e_k^{l,m_1,...,m_k}$, such that*

   $$p \nmid l, \quad m_i = p^{\alpha_i} \cdot r_i, \quad p \nmid r_i, \quad i = 1, 2, ..., k,$$

   the verbal subgroup $e_k^{l,m_1,...,m_k}(G)$ coincides with $\gamma_{s+1}(G)$, where $s = \sum_{i=1}^{k} p^{\alpha_i}$, and $\mathrm{wid}_{e_k^{l,m_1,...,m_k}}(G) = 1$.

In the remaining part of the paper all finitely dimensional matrices will be denoted with lowercase letters, while for the infinite matrices we will use the uppercase letters. For every matrix $a \in \mathrm{UT}(n, K)$ (or $A \in \mathrm{UT}(\infty, K)$) and $m \leq n$ by $a[m]$ (and $A[m]$, respectively) we denote the top-left block of size $m \times m$ of matrix a (or A). The identity matrices of the groups $\mathrm{UT}(n, K)$ and $\mathrm{UT}(\infty, K)$ will be denoted by e_n and E, respectively. Every unitriangular matrix $a \in \mathrm{UT}(n, K)$ may be written as a sum:

$$a = e_n + \sum_{1 \leq i < j \leq n} a_{ij} e_{ij},$$

where e_{ij} denotes elementary matrix of size equal to the size of a, which has 1 in the place (i, j) and zeros elsewhere (infinite elementary matrices will be denoted by E_{ij}).

In Section 2 we discuss verbal subgroups in the groups of infinite dimensional matrices $\mathrm{UT}(\infty, K)$ and $\mathrm{UT}_f(\infty, K)$, using the notion of an inverse and direct systems of groups. Section 3 contains some technical results on the commutator and power equations in the groups of finite unitriangular matrices. These are stated as few lemmata and propositions, which we further use in Section 4 for the proof of the Theorem.

2. Verbal subgroups in $\mathrm{UT}(\infty, K)$

To begin, we note that the groups of infinite unitriangular matrices $\mathrm{UT}_f(\infty, K)$ and $\mathrm{UT}(\infty, K)$ can be constructed from the groups of finitely dimensional matrices $\mathrm{UT}(n, K)$ using the direct and inverse limits. The notion of direct and inverse limit is commonly used in group theory for constructing new groups from old (see [6], p.64, for detailed introduction).

The groups $\mathrm{UT}(n, K)$, $n \in \mathbb{N}$, constitute an infinite system of matrix groups, which is linearly ordered with respect to the dimension n. For every $i, j \in \mathbb{N}$, $i < j$, the groups $\mathrm{UT}(i, K)$ and $\mathrm{UT}(j, K)$ are homomorphic images of each other via respective projections and embeddings. For instance, the group $\mathrm{UT}(j, K)$ may be mapped onto $\mathrm{UT}(i, K)$ using the projection π_{ji}, which deletes the last $(j - i)$ rows and the last $(j - i)$ columns of the matrix. Hence, the groups $\mathrm{UT}(i, K)$ together with the projections π_{ji} constitute an inverse system of groups:

$$\mathrm{UT}(1, K) \xleftarrow{\pi_{21}} \mathrm{UT}(2, K) \xleftarrow{\pi_{32}} \mathrm{UT}(3, K) \xleftarrow{\pi_{43}} \dots$$

The inverse limit

$$\varprojlim_{i,j \in \mathbb{N}} (\mathrm{UT}(i, K), \pi_{ji})$$

of the inverse system $(\mathrm{UT}(i, K), \pi_{ji})$ coincides with the group of infinite unitriangular matrices $\mathrm{UT}(\infty, K)$. We note that as an inverse limit the group $\mathrm{UT}(\infty, K)$ projects to any of the groups $\mathrm{UT}(i, K)$, $i \in \mathbb{N}$, via the canonical projection Π_i, which deletes all but the first i rows and columns of the infinite matrix $A \in \mathrm{UT}(\infty, K)$:

$$\Pi_i(A) = A[i].$$

On the other hand, one can construct also the direct system of groups using the natural embeddings $\varphi_{ij} : \mathrm{UT}(i,K) \hookrightarrow \mathrm{UT}(j,K)$:

$$\mathrm{UT}(1,K) \xrightarrow{\varphi_{12}} \mathrm{UT}(2,K) \xrightarrow{\varphi_{23}} \mathrm{UT}(3,K) \xrightarrow{\varphi_{34}} \ldots$$

The embedding φ_{ij} maps every matrix a of size i to a block-diagonal matrix $\bar{a}$ of size j, such that $\bar{a}[n] = a$, $\bar{a}_{k,k} = 1$ for all $i < k \le j$ and all other entries of $\bar{a}$ are zeros. The direct limit of the constructed direct system

$$\lim_{\overrightarrow{i}} \big(\,\mathrm{UT}(i,K), \varphi_{i,i+1}\big)$$

is exactly the group of finitary matrices $\mathrm{UT}_f(\infty,K)$. Every matrix $A \in \mathrm{UT}_f(\infty,K)$ differs from E only in a finite block $A[n]$ for some n.

Let W be a set of words. It is known (see [14], p. 5) that if $f : G \longrightarrow H$ is a epimorphism of groups G and H, then the homomorphic image of the verbal subgroup $W(G)$ in G is the verbal subgroup in H, defined by the same generating set of words W, i.e. $f(W(G)) = W(H)$.

If $(G_i, f_i)_{i\in\mathfrak{I}}$ is a direct system of groups with a direct limit group G, then $(W(G_i), \bar{f}_i)_{i\in\mathfrak{I}}$, where $\bar{f}_i$ is the restriction of the embedding f_i to the verbal subgroup $W(G_i)$, is also a direct system of groups. In this case we have

Lemma 1. *Let $(G_i, f_i)_{i\in\mathfrak{I}}$ be a direct system of groups defined above, and let W be a set of words. Then*

$$\lim_{\overrightarrow{i}} \big(W(G_i), \bar{f}_i\big)_{i\in\mathfrak{I}} = W(G).$$

Moreover, if $wid_W(G_i) = n$ for all $i \in \mathfrak{I}$, then $wid_W(G) = n$.

The proof may be found in [2].

Now, let $(G_i, f_{ij})_{i,j\in\mathfrak{I}}$ be an inverse system of groups with projections f_{ij} and the inverse limit group G. Since $f_{ij}(W(G_i)) = W(f_{ij}(G_i)) = W(G_j)$, then the verbal subgroups $W(G_i)$, $i \in \mathfrak{I}$, together with the respective projections f_{ij}, restricted to $W(G_i)$, constitute an inverse spectrum of groups. Hence, one may expect that

$$\lim_{\overleftarrow{i}} \left(W(G_i), f_{ij}\right)_{i\in\mathfrak{I}} = W(G).$$

However, in general this equality may not hold. The following lemma provides the necessary and sufficient condition for the equality to be satisfied.

Lemma 2. *Let $(G_i, f_{ij})_{i\in\mathfrak{I}}$ be an inverse system of groups with the limit group G, and let W be a set of words. Then $wid_W(G) = n$ and*

$$\lim_{\overleftarrow{i}} \left(W(G_i), f_{ij}\right)_{i\in\mathfrak{I}} = W(G)$$

if and only if $wid_W(G_i) \le n$ for every $i \in \mathfrak{I}$.

Proof. Assume first that $wid_W(G) = n$ and $\lim\limits_{\substack{\leftarrow \\ i}} (W(G_i), f_{ij})_{i\in\mathfrak{I}} = W(G)$. Let $\Pi_i : G \longrightarrow G_i$, $i \in \mathfrak{I}$, be the canonical projections of the inverse limit G onto the constituent groups G_i. Then for every $i \in \mathfrak{I}$ we have $\Pi_i(W(G)) = W(G_i)$ and, since every element $g \in W(G)$ is a product

$$g = w_1(g_1^1, g_2^1, ..., g_{k(1)}^1)w_2(g_1^2, g_2^2, ..., g_{k(2)}^2)...w_n(g_1^n, g_2^n, ..., g_{k(n)}^n)$$

of n values from G^W, where $w_1, ...w_n \in W$ and $g_j^s \in G$, then we have:

$$\begin{aligned}\Pi_i(g) &= \Pi_i\left(w_1(g_1^1, g_2^1, ..., g_{k(1)}^1)w_2(g_1^2, g_2^2, ..., g_{k(2)}^2)...w_n(g_1^n, g_2^n, ..., g_{k(n)}^n)\right) = \\ &= w_1(\Pi_i(g_1^1), \Pi_i(g_2^1), ..., \Pi_i(g_{k(1)}^1))...w_n(\Pi_i(g_1^n), \Pi_i(g_2^n), ..., \Pi_i(g_{k(n)}^n)) = \\ &= w_1(\bar{g}_1^1, \bar{g}_2^1, ..., \bar{g}_{k(1)}^1)...w_n(\bar{g}_1^n, \bar{g}_2^n, ..., \bar{g}_{k(n)}^n),\end{aligned}$$

where $\bar{g}_j^s \in G_i$ for all s and j. That is, for every $i \in \mathfrak{I}$ we have $wid_W(G_i) \leq n$.

Now let $H = \lim\limits_{\substack{\leftarrow \\ i}} (W(G_i), f_{ij})_{i\in\mathfrak{I}}$. We will show both inclusions $H \subseteq G$ and $H \supseteq G$. First, take $h \in H$ and consider its image under the natural projections $f_i : G \to G_i$. Obviously $f_i(h) = h_i \in W(G_i)$. Since $wid_W(G) = n$, then for every $i \in \mathfrak{I}$ one has

$$h_i = \prod_{k=1}^{n} w_k(g_{i_{1,k}}^{(i)}, g_{i_{2,k}}^{(i)}, ..., g_{i_{t(k)},k}^{(i)}),$$

and $f_{ij}(h_i) = h_j$. Let g_j be the element in G such that $f_i(g_j) = g_j^{(i)}$ for all $i \in \mathfrak{I}$, $k = 1, ..., n$ and $j \in \{i_{1,k}, i_{2,k}, ..., i_{t(k),k}\}$. Then

$$h = \prod_{k=1}^{n} w_k(g_{i_{1,k}}, g_{i_{2,k}}, ..., g_{i_{t(k)},k}) \in W(G).$$

Moreover, $wid_W(G) = n$.

Conversely, take $g \in W(G)$:

$$g = \prod_{k=1}^{s} w_k(g_{i_{1,k}}, g_{i_{2,k}}, ..., g_{i_{t(k)},k}).$$

Then

$$f_i(g) = \prod_{k=1}^{s} w_k(g_{i_{1,k}}^{(i)}, g_{i_{2,k}}^{(i)}, ..., g_{i_{t(k)},k}^{(i)}) \in W(G_i)$$

for every $i \in \mathfrak{I}$. Hence $g \in H$ and the proof is complete.

□

The above lemma shows the difference between the inverse and direct limits of verbal subgroups. The direct limit of a system of verbal subgroups is always a verbal subgroup (generated by the same set of words) and its width is determined by the width of the verbal subgroups in the defining direct system.

To use our Lemma for characterization of verbal subgroups in $\mathrm{UT}(\infty, K)$ we first recall the facts from [2], [3] and [4] on verbal subgroups in groups of finite and finitary unitriangular matrices. We state them below in a single lemma:

Lemma 3. *Let G be one of the groups $\mathrm{UT}(n, K)$, $n \geq 2$, or $\mathrm{UT}_f(\infty, K)$. Then*

1. *$\omega_k(G) = \gamma_k(G)$ and $wid_{\omega_k}(G) = 1$;*
2. *$e_k(G) = \gamma_k(G)$ and $wid_{e_k}(G) = 1$;*
3. *If char $K = 0$ and $m \neq 0$, then $x^m(G) = G$ and $wid_{x^m}(G) = 1$;*
4. *If char $K = p \neq 2$ and $m \neq 0$, $p \nmid m$, then $x^m(G) = G$ and $wid_{x^m}(G) = 1$;*
5. *If char $K = p \neq 2$ and $m \neq 0$, $p^\alpha | m$, $p^{\alpha+1} \nmid m$, then $x^m(G) = \gamma_{p^\alpha}(G)$ and $wid_{x^m}(G) = 2$.*

Note that in all cases in Lemma 3 the width of respective verbal subgroups is uniformly bounded (either by 1 or 2). Hence, by Lemma 2, the following proposition holds.

Proposition 1.

1. *$\gamma_m(\mathrm{UT}(\infty, K)) = \mathrm{UT}(\infty, m-1, K)$;*
2. *$\omega_k(\mathrm{UT}(\infty, K)) = \gamma_k(\mathrm{UT}(\infty, K))$ and $wid_{\omega_k}(\mathrm{UT}(\infty, K)) = 1$;*
3. *$e_k(\mathrm{UT}(\infty, K)) = \gamma_k(\mathrm{UT}(\infty, K))$ and $wid_{e_k}(\mathrm{UT}(\infty, K)) = 1$;*
4. *If char $K = 0$ and $m \neq 0$, then $x^m(\mathrm{UT}(\infty, K)) = \mathrm{UT}(\infty, K)$ and $wid_{x^m}(\mathrm{UT}(\infty, K)) = 1$;*
5. *If char $K = p > 0$ and $m \neq 0$, $p \nmid m$, then $x^m(\mathrm{UT}(\infty, K)) = \mathrm{UT}(\infty, K)$ and $wid_{x^m}(\mathrm{UT}(\infty, K)) = 1$;*
6. *If char $K = p > 0$ and $m \neq 0$, $p^\alpha | m$, $p^{\alpha+1} \nmid m$, then $x^m(\mathrm{UT}(\infty, K)) = \gamma_{p^\alpha}(\mathrm{UT}(\infty, K))$ and $wid_{x^m}(\mathrm{UT}(\infty, K)) = 2$.*

3. Powers and power commutators in $\mathrm{UT}(n, K)$

3.1. Power outer commutators in the case char$K = 0$.

We begin with observations on powers of finite unitriangular matrices. If the characteristic of field K is zero, then the group $\mathrm{UT}(n, K)$ is divisible, i.e. every matrix in $\mathrm{UT}(n, K)$ has k-th roots for all $k \geq 2$ (see Lemma 3). Hence, in this situation as a straightforward consequence we obtain:

Proposition 2. *Let char $K = 0$. Then*

$$\omega_k^{m_1,\ldots,m_k}(\mathrm{UT}(n, K)) = \omega_k(\mathrm{UT}(n, K)) = \gamma_k(\mathrm{UT}(n, K))$$

and $wid_{\omega_k^{m_1,\ldots,m_k}}(\mathrm{UT}(n, K)) = 1$.

Proof. Let a be an arbitrary matrix in $\gamma_k(\mathrm{UT}(n, K)) = \mathrm{UT}(n, k-1, K)$. By statement (1) in Lemma 3 there exist matrices $b_1, \ldots, b_k \in \mathrm{UT}(n, K)$ such that $a = \omega_k(b_1, \ldots, b_k)$. Now, by statement (3) in Lemma 3 we deduce that all b_i, $i = 1, \ldots, k$, are arbitrary powers. In particular one can find matrices $c_1, \ldots, c_k \in \mathrm{UT}(n, K)$ such that $b_i = c_i^{m_i}$ for $i = 1, \ldots, k$. Then obviously $a = \omega_k(c_1^{m_1}, \ldots, c_k^{m_k})$, i.e. a is a value of the power commutator $\omega_k^{m_1,\ldots,m_k}$. Thus $\gamma_k(\mathrm{UT}(n, K)) \subseteq \omega_k^{m_1,\ldots,m_k}(\mathrm{UT}(n, K))$. As the reverse inclusion is obvious, we have completed the proof. □

3.2. Power outer commutators in the case char$K = p$

In the case of matrices over a field of characteristic $p > 0$, one must take care of the powers, which are divisible by p. Lemma 3 shows, that in this case there exist matrices, which cannot be written as such a a power. On the other hand, there are matrices, for which the representation as an arbitrary power in $\mathrm{UT}(n, K)$ is possible. We characterize them in the following proposition.

Proposition 3. *Let char $K = p > 0$ and $m = p^\alpha$. Then every matrix $c \in \mathrm{UT}(n, m-1, K)$, such that $rank(c - e_n) = n - m$ is the m-th power in $\mathrm{UT}(n, K)$. In particular, there exists matrix $a \in \mathrm{UT}(n, K)$ such that $c = a^m$ and $rank(a - e_n) = n - 1$.*

Proof. Let c be a matrix in $\mathrm{UT}(n, m-1, K)$, such that $rank(c - e_n) = n - m$. If $n \leq m$ then $c = e_n$ (see Lemma 3) and for every matrix $a \in \mathrm{UT}(n, K)$ we have $c = a^m$. In particular, one may choose a satisfying the condition $rank(a - e_n) = n - 1$.

Now, assume that the statement of our proposition holds for all $n \leq N$. Take $\bar{c} \in \mathrm{UT}(N+1, m-1, K)$ to be the matrix

$$\bar{c} = \begin{pmatrix} c & \mathbf{c} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $c \in \mathrm{UT}(N, m-1, K)$, $\mathbf{c}^T = (c_1, c_2, ..., c_{N-m+1}, 0, ..., 0) \in K^N$ and $\mathbf{0}$ is a zero vector in K^N. From the inductive assumption one finds matrix $a \in \mathrm{UT}(N, K)$ with $a^m = c$ and $rank(a - e_N) = N - 1$. Note that the latter condition is equivalent to saying that $\prod_{i=1}^{N-1} a_{i,i+1} \neq 0$.

Now, let $\bar{a} \in \mathrm{UT}(N+1, K)$ be defined as follows:

$$\bar{a} = \begin{pmatrix} a & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix},$$

with $\mathbf{a}^T = (a_1, a_2, ..., a_N) \in K^N$. We will show that it is possible to choose $\mathbf{a}$ such that $\bar{c} = \bar{a}^m$ and $rank(\bar{a} - e_{N+1}) = N$.

We first observe that

$$\bar{a}^m = \begin{pmatrix} a^m & d_m(a)\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $d_m(a) = e_N + a + a^2 + ... + a^{m-1}$. We note that since m is the power of the field characteristic p, then $d_m(a)$ is not invertible. We calculate the entries of matrix $d_m(a)$:

$$[d_m(a)]_{ij} = \begin{cases} 0, & \text{if} \quad 1 \leq j < i + m - 1, \\ \sum_{t_1=i+1}^{j-1} \sum_{t_2=i+1}^{t_1-1} ... \sum_{t_{m-2}=i+1}^{t_{m-3}-1} a_{i,t_{m-2}} \cdot a_{t_{m-2},t_{m-3}} \cdot ... \cdot a_{t_2,t_1} \cdot a_{t_1,j}, & \text{otherwise.} \end{cases}$$

Thus, in particular $[d_m(a)]_{i,i+m-1} = a_{i,i+1} \cdot a_{i+1,i+2} \cdot ... \cdot a_{i+m-2,i+m-1} \neq 0$ for every $i = 1, ..., N - m + 1$. This implies that $rank(d_m(a)) = N - m + 1$ and hence, for every $\mathbf{c}^T = (c_1, c_2, ..., c_{N-m+1}, 0, ..., 0) \in K^N$ the equation

$$d_m(a) \cdot \mathbf{a} = \mathbf{c}$$

has a solution in terms of **a**. Moreover, since $(d_m(a))_{N-m+1,N} \neq 0$ and $c_{N-m+1} \neq 0$, we have $a_N \neq 0$. This implies that $\bar{c} = \bar{a}^m$ with $rank(\bar{a} - e_{N+1}) = N$ and by induction the proof is complete. □

Now we discuss the values of commutators of powers in $\mathrm{UT}(n, K)$. We will need the following result from [3]:

Lemma 4. *Let K be an arbitrary field. For every matrix $c \in \mathrm{UT}(n, i+j+1, K)$ there exist matrices $a \in \mathrm{UT}(n, i, K)$ and $b \in \mathrm{UT}(n, j, K)$ such that $rank(a - e_n) = n - i - 1$, $rank(b - e_n) = n - j - 1$ and $c = [a, b]$.*

Merging the above results, we obtain:

Proposition 4. *Let char $K = p > 0$ and let $\omega_k^{m_1,\dots,m_k}$ be a power outer commutator, such that $m_i = p^{\alpha_i} \cdot r_i$, $\alpha_i \geq 0$ and $p \nmid r_i$ for $i = 1, 2, \dots, k$. Then*

$$\omega_k^{m_1,\dots,m_k}(\mathrm{UT}(n, K)) = \gamma_s(\mathrm{UT}(n, K)),$$

where $s = \sum_{i=1}^{k} p^{\alpha_i}$ and $wid_{\omega_k^{m_1,\dots,m_k}}(\mathrm{UT}(n, K)) = 1$.

Proof. Observe that for every $a_1, a_2, \dots, a_k \in \mathrm{UT}(n, K)$ the value of $\omega_k^{m_1,\dots,m_k}(a_1, a_2, \dots, a_k)$ lies in $\gamma_s(\mathrm{UT}(n, K)) = \mathrm{UT}(n, s-1, K)$, with s defined in the statement of the proposition. This can be seen by induction. It is clear that if $k = 2$ we have

$$\omega_2^{m_1,m_2}(a_1, a_2) = [a_1^{m_1}, a_2^{m_2}]$$

and, by Lemma 3, $\omega_2^{m_1,m_2}(a_1, a_2)$ is an element of

$$[\mathrm{UT}(n, p^{\alpha_1} - 1, K), \mathrm{UT}(n, p^{\alpha_2} - 1, K)] = \mathrm{UT}(n, p^{\alpha_1} + p^{\alpha_2} - 1, K).$$

Now, if we assume that

$$\omega_j^{m_1,\dots,m_j}(a_1, a_2, \dots, a_j) \in \mathrm{UT}(n, \sum_{i=1}^{j} p^{\alpha_i} - 1, K)$$

for all $j \leq k$ and consider the values of the word $\omega_{k+1}^{m_1,\dots,m_{k+1}}$ in $\mathrm{UT}(n, K)$, then we can write any value of the word as a commutator of two power outer commutators, each of them of weight not greater than k, say i and j respectively. Hence, due to our inductive assumption we obtain:

$$\begin{aligned}\omega_{k+1}^{m_1,\dots,m_{k+1}}(\mathrm{UT}(n, K)) &= [\omega_i^{m_1,\dots,m_i}(\mathrm{UT}(n, K)), \omega_j^{m_{i+1},\dots,m_j}(\mathrm{UT}(n, K))] = \\ &= [\mathrm{UT}(n, \sum_{t=1}^{i} p^{\alpha_t} - 1, K), \mathrm{UT}(n, \sum_{t=i+1}^{j} p^{\alpha_t} - 1, K)] \subseteq \mathrm{UT}(n, \sum_{t=1}^{i+j} p^{\alpha_t} - 1, K),\end{aligned}$$

where $i + j = k + 1$. Indeed, by induction we have

$$\omega_k^{m_1,\dots,m_k}(\mathrm{UT}(n, K)) \subseteq \mathrm{UT}(n, s-1, K).$$

Now, for the reverse inclusion we take an arbitrary matrix $c \in \mathrm{UT}(n, s-1, K)$. Let $\omega_k^{m_1,...,m_k} = [\omega_i^{m_1,...,m_i}, \omega_j^{m_{i+1},...,m_k}]$, where $i+j=k$. From what we proved above we have that

$$\omega_i^{m_1,...,m_i}(\mathrm{UT}(n,K)) \subseteq \mathrm{UT}(n, s_i-1, K),$$
$$\omega_j^{m_{i+1},...,m_k}(\mathrm{UT}(n,K)) \subseteq \mathrm{UT}(n, s_j-1, K),$$

where $s_i = \sum_{t=1}^{i} p^{\alpha_t}$, $s_j = \sum_{t=i+1}^{k} p^{\alpha_t}$. Now, applying Lemma 4 we write $c = [a,b]$, where $a \in \mathrm{UT}(n, s_i-1, K)$, $b \in \mathrm{UT}(n, s_j-1, K)$ and $rank(a-e_n) = n-s_i$, $rank(b-e_n) = n-s_j$. Then we repeat this reasoning for matrices a and b, and write each of them as a value of a commutator of outer commutators of weights smaller than i and j, respectively. We perform this step repeatedly until the weight of every outer commutator is 1. This way we find matrices $a_1,...,a_k$, $a_i \in \mathrm{UT}(n, p^{\alpha_i}-1, K)$, $rank(a_i-e_n) = n-p^{\alpha_i}$, $i=1,...,k$, such that:

$$c = [a_1, a_2, ..., a_k].$$

Note that matrices a_i satisfy the assumptions of Proposition 3. Hence every matrix a_i is a power $a_i = b_i^{m_i}$ of certain matrix $b_i \in \mathrm{UT}(n,K)$. Thus

$$c = [b_1^{m_1}, b_2^{m_2}, ..., b_k^{m_k}] = \omega_k^{m_1,...,m_k}(b_1,...,b_k),$$

i.e. $\mathrm{UT}(n, s-1, K) \subseteq \omega_k^{m_1,...,m_k}(\mathrm{UT}(n,K))$. This completes the proof. □

3.3. Power Engel commutators

For the power Engel commutators we will need a stronger result than those stated in [4]. The following lemma provides a useful technical observation for fields with either positive or zero characteristic.

Lemma 5. *Assume K is a field with $charK \neq 2$. Let $b \in \mathrm{UT}(n,i,K)$ be a matrix, such that $rank(b-e_n) = n-i-1$. Then for every matrix $c \in \mathrm{UT}(n, i+j+1, K)$, there exists a matrix $a \in \mathrm{UT}(n,j,K)$, such that $c=[a,b]$.*

Proof. We prove the result by induction on the matrix size n. We start with the smallest n possible, i.e. $n = i+j+2$. Then the matrix $c \in \mathrm{UT}(n, i+j+1, K)$ is the identity matrix and for every matrices $a \in \mathrm{UT}(n,j,K)$ and $b \in \mathrm{UT}(n,i,K)$ we have $[a,b] = e_n = c$. Thus, in particular, we may choose b such that $rank(b-e_n) = n-i-1$.

Now let us assume that the lemma holds for all matrices of sizes not greater than n. Let us take $\bar{c} \in \mathrm{UT}(n+1, i+j+1, K)$ and $\bar{b} \in \mathrm{UT}(n+1, i, K)$, such that $rank(b-e_{n+1}) = n-i$. We represent matrices $\bar{b}$ and $\bar{c}$ as:

$$\bar{c} = \begin{pmatrix} c & \mathbf{c} \\ \mathbf{0} & 1 \end{pmatrix}, \qquad \bar{b} = \begin{pmatrix} b & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $\mathbf{c}^T = (c_1, c_2, ..., c_{n-i-j-1}, 0, ..., 0) \in K^n$, $\mathbf{0}$ is a zero vector from K^n. Observe that due to our assumptions on $\bar{b}$, we have $\mathbf{b}^T = (b_1, ..., b_{n-i}, 0, ..., 0) \in K^n$ with $b_{n-i} \neq 0$. Now, let $\bar{a} \in \mathrm{UT}(n+1, j, K)$ be the matrix:

$$\bar{a} = \begin{pmatrix} a & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $a \in \mathrm{UT}(n, j, K)$ and $\mathbf{a}^T = (a_1, ..., a_{n-j}, 0, ..., 0) \in K^n$. Then we have:

$$[\bar{a}, \bar{b}] = \begin{pmatrix} [a,b] & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix},$$

where $\mathbf{x} = a^{-1}(b^{-1} - e_n)\mathbf{a} + a^{-1}b^{-1}(a - e_n)\mathbf{b}$. By our inductive assumption we can choose $a \in \mathrm{UT}(n, j, K)$ such that $c = [a, b]$. Then we have:

$$rank(a - e_n) \leq n - j - 1,$$
$$rank(b^{-1} - e_n) = rank(b - e_n) = n - i - 1.$$

Let us denote $f = a^{-1}b^{-1}(a - \mathbf{1}_n)$ and $d = a^{-1}(b^{-1} - \mathbf{1}_n)$. Since multiplication by invertible matrices does not affect the matrix rank, we have

$$rank(f) \leq n - j - 1,$$
$$rank(d) = n - i - 1.$$

For the equality $\bar{c} = [\bar{a}, \bar{b}]$ to hold it is necessary that

$$\mathbf{c} = d\mathbf{a} + f\mathbf{b}.$$

Since f and $\mathbf{b}$ are already determined, we observe that

$$f\mathbf{b} = \mathbf{f} = (f_1, f_2, ..., f_{n-j-i-1}, 0, ..., 0)^T$$

and we have to solve the system:

$$\mathbf{w} = d\mathbf{a},$$

where $\mathbf{w} = \mathbf{c} - \mathbf{f} = (w_1, w_2, ..., w_{n-i-j}, 0, ..., 0)^T$. Now, considering the rank of the augmented matrix of the system

$$rank(d|\mathbf{w}) = \max\{n - i - 1, n - i - j - 1\} = n - i - 1$$

we have that $rank(d|\mathbf{w}) = rank(d)$ hence the system has a solution $\mathbf{a}$. It is clear that $\bar{a} \in \mathrm{UT}(n+1, n-j, K)$ and the lemma follows by induction. □

Corollary 1. *In the case of* $\mathrm{char} K = 0$*, for arbitrary powers* $l, m_1, ..., m_k \in \mathbb{Z} \setminus \{0\}$ *and an arbitrary matrix* $c \in \mathrm{UT}(n, k, K)$ *there exist matrices* $a, b \in \mathrm{UT}(n, K)$ *such that* $c = e_k^{l,m_1,...,m_k}(a, b)$*. In other words*

$$e_k^{l,m_1,...,m_k}(\mathrm{UT}(n, K)) = \gamma_{k+1}(\mathrm{UT}(n, K))$$

and the width of any verbal subgroup generated by a power Engel word in $\mathrm{UT}(n, K)$ *is equal to 1.*

Proof. Let $b \in \mathrm{UT}(n, K)$ be a matrix such that $rank(b - e_n) = n - 1$. Then it is clear that $rank(b^m - e_n) = n - 1$ for an arbitrary power $m \neq 0$. Now let $c \in \mathrm{UT}(n, k, K)$ and let $l, m_1, ..., m_k \in \mathbb{Z} \setminus \{0\}$ be arbitrary powers. Then, by Lemma 5, there exists a matrix $a_1 \in \mathrm{UT}(n, k-1, K)$ such that $c = [a_1, b^{m_k}]$. Then again we find matrix $a_2 \in \mathrm{UT}(n, k-2, K)$ such that $a_1 = [a_2, b^{m_{k-1}}]$ and, in the i-th step we find $a_i \in \mathrm{UT}(n, k-i, K)$ such that $a_{i-1} = [a_i, b^{m_{k-i}}]$. After k steps we come to the conclusion that $c = [a_k, b^{m_1}, ..., b^{m_k}]$. Thus if $a = a_k$ then $c = e_k^{l,m_1,...,m_k}(a, b)$. □

Now we discuss the case when $charK > 0$. We prove the following:

Proposition 5. *Let K be a field such that $charK = p \neq 2$. Then for every power Engel commutator $e_k^{l,m_1,...,m_k}$, such that*

$$p \nmid l, \quad m_i = p^{\alpha_i} \cdot r_i, \quad p \nmid r_i, \quad i = 1, 2, ..., k,$$

the verbal subgroup $e_k^{l,m_1,...,m_k}(\mathrm{UT}(n, K))$ coincides with $\gamma_{s+1}(\mathrm{UT}(n, K))$, where $s = \sum_{i=1}^{k} p^{\alpha_i}$, and $wid_{e_k^{l,m_1,...,m_k}}(\mathrm{UT}(n, K)) = 1$.

Proof. Let us choose matrix $b = e_n + \sum_{i=1}^{n-1} e_{i,i+1} \in \mathrm{UT}(n, K)$. It is clear that

$$\begin{array}{rcll} rank(b - e_n) & = & n - 1, & \\ rank(b^w - e_n) & = & n - p^\alpha, & \text{where } w = p^\alpha \cdot r, \quad p \nmid r. \end{array} \tag{1}$$

The detailed explanation of the latter equality the reader finds in Section 3.1 in [3].

Now, let c be an arbitrary matrix in $\gamma_{s+1}(\mathrm{UT}(n, K)) = \mathrm{UT}(n, s, K)$, where s is defined as above. By (1) we see that b^{m_i} for $i = 1, ..., k$ satisfy the assumptions of Lemma 5 and hence there exist matrices

$$a_i \in \mathrm{UT}(n, \sum_{j=1}^{k-i} p^{\alpha_j}, K), \quad i = 1, ..., k-1,$$

and a matrix $a_k \in \mathrm{UT}(n, K)$ such that

$$c = [a_1, b^{m_k}], \quad a_i = [a_{i+1}, b^{m_{k-i}}], \quad i = 1, ..., k-1.$$

Hence we obtain

$$c = [a_k, b^{m_1}, b^{m_2}, ..., b^{m_k}].$$

Since p is not a divisor of l, then by Lemma 3 there exists a matrix $a \in \mathrm{UT}(n, K)$ such that $a^l = a_k$. Thus $c = e_k^{l,m_1,...,m_k}(a, b)$ and the Proposition follows.

□

4. Proof of the Theorem

The Propositions 2, 4, 5 and Corollary 1, which we proved in Section 3, combine together to the statements of our Theorem for groups of finitely dimensional matrices $\mathrm{UT}(n, K)$. We proved that for any characteristic of K different than 2, the width of the verbal subgroups under investigation is regardless of n is equal to 1 in all of the groups $UT(n, K)$. Thus, by Lemma 1, our theorem holds also for the finitary groups $\mathrm{UT}_f(\infty, K)$:

$$\begin{array}{rcl} \omega_k^{m_1,...,m_k}(\mathrm{UT}_f(\infty, K)) & = & \lim\limits_{\overrightarrow{i}} \left(\omega_k^{m_1,...,m_k}(\mathrm{UT}(i, K)), \varphi_{i,i+1}\right)_{i\in\mathbb{N}} = \\ & = & \lim\limits_{\overrightarrow{i}} (\gamma_s(\mathrm{UT}(i, K)), \varphi_{i,i+1})_{i\in\mathbb{N}} = \gamma_s(\mathrm{UT}_f(\infty, K)) \end{array}$$

and

$$\begin{aligned} e_k^{l,m_1,\ldots,m_k}(\mathrm{UT}_f(\infty,K)) &= \lim_{\overrightarrow{i}} \left(e_k^{l,m_1,\ldots,m_k}(\mathrm{UT}(i,K)),\varphi_{i,i+1}\right)_{i\in\mathbb{N}} = \\ &= \lim_{\overrightarrow{i}} \left(\gamma_{s+1}(\mathrm{UT}(i,K)),\varphi_{i,i+1}\right)_{i\in\mathbb{N}} = \gamma_{s+1}(\mathrm{UT}_f(\infty,K)), \end{aligned}$$

where $s=k$ if K is a field of characteristic 0 and $s=\sum\limits_{i=1}^{k} p^{\alpha_i}$, if $char K = p \neq 2$. Also, since 1 is the uniform bound for the width of every verbal subgroup in $UT(n,K)$, $n\in\mathbb{N}$, generated by the power outer commutator or a power Engel commutator, we may apply Lemma 2 to obtain that:

$$\begin{aligned} \omega_k^{m_1,\ldots,m_k}(\mathrm{UT}(\infty,K)) &= \lim_{\overleftarrow{i}} \left(\omega_k^{m_1,\ldots,m_k}(\mathrm{UT}(i,K)),\pi_{ij}\right)_{i\in\mathbb{N}} = \\ &= \lim_{\overleftarrow{i}} \left(\gamma_s(\mathrm{UT}(i,K)),\pi_{ij}\right)_{i\in\mathbb{N}} = \gamma_s(\mathrm{UT}(\infty,K)) \end{aligned}$$

and

$$\begin{aligned} e_k^{l,m_1,\ldots,m_k}(\mathrm{UT}(\infty,K)) &= \lim_{\overleftarrow{i}} \left(e_k^{l,m_1,\ldots,m_k}(\mathrm{UT}(i,K)),\pi_{ij}\right)_{i\in\mathbb{N}} = \\ &= \lim_{\overleftarrow{i}} \left(\gamma_{s+1}(\mathrm{UT}(i,K)),\pi_{ij}\right)_{i\in\mathbb{N}} = \gamma_{s+1}(\mathrm{UT}(\infty,K)), \end{aligned}$$

where $s=k$ if K is a field of characteristic 0 and $s=\sum\limits_{i=1}^{k} p^{\alpha_i}$, if $char K = p \neq 2$. Thus the theorem follows. □.

We conclude with a straightforward consequence of the Theorem for solvability of power commutator equations in groups of unitriangular matrices. Namely, we have:

Corollary 2. *Let K be a field of characteristic not equal to 2 and let G be one of the groups $\mathrm{UT}(n,K)$, $\mathrm{UT}_f(\infty,K)$ or $\mathrm{UT}(\infty,K)$.*

1. *If $char K = 0$ then the equation $\omega_k^{m_1,\ldots,m_k}(x_1,x_2,\ldots,x_k)=a$ has solution $x_1,\ldots,x_k\in G$ if and only if $a\in\gamma_k(G)$;*
2. *If $char K = 0$ then the equation $e_k^{l,m_1,\ldots,m_k}(x,y)=a$ has solution $x,y\in G$ if and only if $a\in\gamma_{k+1}(G)$;*
3. *If $char K = p$ then the equation $\omega_k^{m_1,\ldots,m_k}(x_1,x_2,\ldots,x_k)=a$ has solution $x_1,\ldots,x_k\in G$ if and only if $a\in\gamma_s(G)$, where $s=\sum\limits_{i=1}^{k} p^{\alpha_i}$;*
4. *If $char K = p$ then the equation $e_k^{l,m_1,\ldots,m_k}(x,y)=a$ has solution $x,y\in G$ if and only if $a\in\gamma_{s+1}(G)$, where $s=\sum\limits_{i=1}^{k} p^{\alpha_i}$.*

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