Finitely Generated G' and Positive Laws

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Abstract

We conjecture that a finitely generated relatively free group G has a finitely generated commutator subgroup G' if and only if G satisfies a positive law. We confirm this conjecture for groups G in the large class, containing all residually finite and all soluble groups.

Let u, v be different words in a free semigroup generated by $X = \{x_1, x_2, ...\}$. A group G satisfies n-variable positive law $u(x_1, ..., x_n) = v(x_1, ..., x_n)$, if under each substitution $X \to G$, the equality $u(g_1, ..., g_n) = v(g_1, ..., g_n)$ holds. We denote: $\mathfrak{N}_c$ – the variety of all nilpotent groups of nilpotency class c,
$\mathfrak{S}_n$ – the variety of all soluble groups of solubility class n,
$\mathfrak{B}_e$ – the Restricted Burnside variety of exponent e, i.e. the variety generated by all finite groups of exponent e. All groups in $\mathfrak{B}_e$ are locally finite of exponent e. The existence of such varieties for each positive integer e follows from the positive solution of the Restricted Burnside Problem and relies on classification of finite simple groups (see [6]).

Every finitely generated group satisfying positive law has finitely generated commutator subgroup ([1], p. 514). The converse is not true in general, for example the group $G = \langle x \rangle_2 * \langle y \rangle_3$ has no laws while G' is finitely generated. We conjecture that if G is a finitely generated *relatively free* group then G' is finitely generated if and only if G satisfies a positive law. We confirm the conjecture for groups in the large class $\mathcal{C}$, introduced in [1] as a sum $\mathcal{C} = \cup_n \Delta_n$, where Δ_1 is the class of groups contained in all finite products of varieties $\mathfrak{V}_1\mathfrak{V}_2...\mathfrak{V}_m$, where $\mathfrak{V}_i$ is either $\mathfrak{S}_d$ or $\mathfrak{B}_e$ for various d, e, and

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$\Delta_{n+1} = \{\text{groups, locally in } \Delta_n\} \cup \{\text{groups, residually in } \Delta_n\}.$

We note that Δ_1 contains all nilpotent-by-finite groups.

$\{\mathbf{G'f.g.}\}$ denotes the class of all finitely generated, relatively free groups with finitely generated commutator subgroups,
PL denotes the class of finitely generated, relatively free groups, which satisfy positive laws. Inclusion $\{\mathbf{G'f.g.}\} \supseteq \mathbf{PL}$ follows from [1].

Our main result here is that $\{\mathbf{G'f.g.}\} \cap \mathcal{C}$ consists of nilpotent-by-finite groups and coincides with $\mathbf{PL} \cap \mathcal{C}$.

Lemma 1 *Let G be in $\{\mathbf{G'f.g.}\}$. Then every derived subgroup in any finitely generated group $H \in var\, G$ is finitely generated.*

Proof Let G be freely generated by $g_1, g_2, ..., g_n$, $n > 1$. If we map all generators, but g_1, g_2 into 1, then the finitely generated subgroup $\langle g_1, G' \rangle$ has an image $\langle g_1 \rangle [\langle g_1 \rangle, \langle g_2 \rangle] = \langle g_1^{\langle g_2 \rangle} \rangle$, which also is finitely generated. Since G is relatively free, it follows that in any finitely generated group $H \in var\, G$ for all $a, b \in H$, the subgroup $\langle a^{\langle b \rangle} \rangle$ is finitely generated. Then by ([4], p. 1421), every derived subgroup in H is finitely generated. □

Lemma 2 *Let G be in $\{\mathbf{G'f.g.}\}$. Then every finitely generated soluble group in $var\, G$ is nilpotent-by-finite.*

Proof It is enough to show that if G itself is soluble then G is nilpotent-by-finite. The group G is isomorphic to F/V, where F is a finitely generated free group and V – a verbal subgroup. If there exists p such that $V \subseteq F''(F')^p$, then $G/G''(G')^p \cong F/F''(F')^p$ has an infinitely generated commutator subgroup. So G also has an infinitely generated commutator subgroup, which contradicts the assumption. Hence for some n and for all p we have $F^{(n)} \subseteq V \nsubseteq F''(F')^p$. Then by ([2], (ii)), G is nilpotent-by-finite. □

Lemma 3 *Let G be in $\{\mathbf{G'f.g.}\}$. If G is residually finite, then G is nilpotent-by-finite.*

Proof As in the proof of Lemma 1, we get that for free generators $g_1, g_2 \in G$ the subgroup $\langle g_1^{\langle g_2 \rangle} \rangle$ is finitely generated by, say, k elements. Since G is relatively free, it follows that for any $a, b \in G$, the subgroup $\langle a^{\langle b \rangle} \rangle$ is generated by at most k elements. Then G has no sections isomorphic to a twisted wreath product $E\,twr_L H$, of an elementary abelian p-group E and a finite cyclic group H, where $|H : L| > k$. Now by Theorem 4 in [7] it follows that G contains a soluble normal subgroup N of finite index. So N is finitely generated and by Lemma 2, N is nilpotent-by-finite. Then G, as a finite extension of N, is also nilpotent-by-finite. $\Box$

Lemma 4 *Let G be in $\{\mathbf{G'f.g.}\}$. Then every section of G, which belongs to a product $\mathfrak{B}_e\mathfrak{S}_d$ for some e, d, is nilpotent-by-finite.*

Proof It is enough to assume that G itself belongs to a product $\mathfrak{B}_e\mathfrak{S}_d$. Then by Lemma 1, $G^{(d)}$ is finitely generated and since $G^{(d)} \in \mathfrak{B}_e$, we get that $G^{(d)}$ is finite. Then the centralizer C of $G^{(d)}$ must have a finite index in G, and hence is finitely generated. Moreover, C is soluble, because $C^{(d+1)} \subseteq [G^{(d)}, C] = 1$. By Lemma 2, C is nilpotent-by-finite. So G, as a finite extension of C, is also nilpotent-by-finite. $\Box$

Theorem 1 *Every group in $\{\mathbf{G'f.g.}\}\cap\mathcal{C}$ is nilpotent-by-finite.*

Proof We show first that every group in G in $\{\mathbf{G'f.g.}\}\cap\Delta_1$ is nilpotent-by-finite. Since G is in $\mathfrak{V}_1\mathfrak{V}_2\ldots\mathfrak{V}_m$, where each variety $\mathfrak{V}_i$ is either soluble or a Restricted Burnside variety, then G has a finite series $1 = N_0 \triangleleft N_1 \triangleleft \ldots N_{m-1} \triangleleft N_m = G$, in which each section N_i/N_{i-1} belongs to $\mathfrak{V}_i$. If a section N_i/N_{i-2} is in $\mathfrak{B}_e\mathfrak{S}_n$ for some e, n, then by Lemma 4, this section belongs to a product of a nilpotent variety and a Restricted Burnside variety. So we can replace (starting from the right) every pair of the type $\mathfrak{B}\mathfrak{S}$ by some pair of the type $\mathfrak{N}\mathfrak{B}$, and obtain, that G belongs to a soluble-by-Restricted Burnside variety $\mathfrak{S}_d\mathfrak{B}_k$ for some d, k. Then, by Lemma 2, G is nilpotent-by-finite. So $\{\mathbf{G'f.g.}\}\cap\Delta_1$ consists of nilpotent-by-finite groups.

If G is a finitely generated group in the class $\mathcal{C}$, then we can see that G is residually in Δ_1. Indeed, if G is in Δ_{n+1}, then G has to be residually in Δ_n. That is G is a subcartesian product of finitely generated quotients $G/N \in \Delta_n$. Then, similarly, each G/N is residually in Δ_{n-1} and hence G is residually in Δ_{n-1}, which implies inductively that G is residually in Δ_1.

Let now G be in $\{\mathbf{G'f.g.}\}\cap\mathcal{C}$, then G is a subcartesian product of its quotients $G/N \in \Delta_1 \cap var\,G$. Since Δ_1 is a union of varieties, G/N is an image of a relatively free group in $\{\mathbf{G'f.g.}\}\cap\Delta_1$, which, as we have proved, is nilpotent-by-finite. Then G/N is nilpotent-by-finite and by [3], G/N is residually finite. It follows that G also is residually finite. Now by Lemma 3, G is nilpotent-by-finite as required. $\Box$

By [1], a finitely generated group $G \in \mathcal{C}$ is nilpotent-by-finite if and only if $G \in \mathbf{PL}$. So we get the required inclusion $\{\mathbf{G'f.g.}\} \cap \mathcal{C} \subseteq \mathbf{PL} \cap \mathcal{C}$ which gives

Corollary 1 *A finitely generated, relatively free group $G \in \mathcal{C}$ satisfies a positive law if and only if its commutator subgroup G' is finitely generated.*

Corollary 2 *Each n-engel group $G \in \mathcal{C}$ satisfies a positive law.*

Let $G \in \mathcal{C}$ be a finitely generated relatively free n-engel group. The law $[x, y, y, ..., y]$ implies that x^{y^n} is in a subgroup generated by $x, x^y, ..., x^{y^{n-1}}$. Then for all $a, b \in G$, the subgroup $\langle a^{\langle b \rangle} \rangle$ is finitely generated and hence G' is finitely generated. Then by Corollary 1, G satisfies a positive law. $\Box$

Question Does there exist a finitely generated, relatively free group G, containing a free nonabelian subsemigroup and with G' finitely generated?

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