Two questions on semigroup laws

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Abstract

B. H. Neumann recently proved some implication for semigroup laws in groups. This may help in solution of a problem posed by G. M. Bergman in 1981.

Let G be a group, and $S \subseteq G$ be a subsemigroup generating G. It is clear that if S is commutative, then G is commutative. The following question is equivalent to the one posed by G. M. Bergman [2], [3].

Question 1 Let S generating G satisfy a law. Must G satisfy the same law?

For some laws the answer is positive [9], [5], [8], [1], however in general the question is open and in opinion of S. V. Ivanov and E. Rips it has a negative answer. All semigroups we consider are cancellative.

Question 2 Let a semigroup law $a = b$ implies a semigroup law $u = v$ in groups. Does the same implication hold in semigroups?

To show implication of laws in semigroups we can use only so-called positive endomorphisms, which map generators to positive words. It is shown in [8] (an example at the end of this paper), that all implications for positive laws of length ≤ 5 which hold in groups, also are valid for semigroups. The fact that the law $x^2y^2x = yx^3y$ implies $xy = yx$ in semigroups (and hence in groups) is proved in [5, p.132].

We show the equivalence of the above Questions.

It is shown in [10], that the law $x^{s+t}y^2x^t = yx^{s+2t}y,\ gcd(s,t) = 1$, implies $xy^2x = yx^2y$ in groups (which is equivalent to $[x, y, x] = 1$ [12]). So if

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there exists a semigroup satisfying $x^{s+t}y^2x^t = yx^{s+2t}y,\ gcd(s,t) = 1$, but not $xy^2x = yx^2y$, the desired counterexample for Question 1 would be found.

Let $a = a(x_1, ..., x_n),\ b = b(x_1, ..., x_n)$ be positive words. A semigroup law $a = b$ is called *balanced* if every x_i has the same exponent sum in a and b. The law is *trivial* if $ab^{-1} = 1$ in F. The law is called *cancelled* if the first (and the last) letters in a and b are different.

Notation

Let F be a free group and $\mathcal{F} \ni 1$ be a free semigroup, both generated by $x_1, x_2, x_3, \ldots$. Words in $\mathcal{F}$ are called positive. We denote:

End^+ – the set of positive endomorphisms which map x_i to positive words,

N_w – a normal End^+-invariant closure of a word w in F,

End – the set of all endomorphisms of the free group F,

V_w – a fully invariant subgroup generated by a word $w \in F$,

$(u, v)^\#$ – the smallest cancellative congruence in $\mathcal{F}$ providing the law $u = v$.

A relatively free cancellative semigroup, defined by the law $u = v$ is isomorphic to $\mathcal{F}/(u, v)^\#$ [8].

We note that if N_w contains a positive word, say x^2yz^4, then it contains x^7 and hence $x^{-1} \in x^6N_w$ implies $F = \mathcal{F}\ mod\ N_w$.

Remark 1 *Since each semigroup with a non-balanced law is a group, we have to consider only balanced non-trivial semigroup laws. Each such a law implies a binary balanced and cancelled law* $A(x, y) = B(x, y)$ [6].

Questions and Results

To formulate the above Questions in terms of normal subgroups we need

Lemma 1 *A semigroup law* $u = v$ *implies* $a = b$ *in semigroups if and only if* $N_{ab^{-1}} \subseteq N_{uv^{-1}}$. *The implication holds in groups if and only if* $V_{ab^{-1}} \subseteq V_{uv^{-1}}$.

Proof The law $u = v$ implies $a = b$ in semigroups if and only if corresponding smallest congruences satisfy $(a, b)^\# \subseteq (u, v)^\#$. If we map $F \to F/N$, then $\mathcal{F}$ is mapped onto $\mathcal{F}/N^\#$, where $N^\#$ is a cancellative congruence in $\mathcal{F}$ defined as: $N^\# = \{(s, t);\ st^{-1} \in N \cap \mathcal{F}\mathcal{F}^{-1}\}$. It is proved in [7], Thm. 2, that $N := N_{uv^{-1}}$ is a smallest normal subgroup such that $N^\# = (u, v)^\#$. So we have

$$(u, v)^\# = \{(s, t);\ st^{-1} \in N_{uv^{-1}} \cap \mathcal{F}\mathcal{F}^{-1}\}. \tag{1}$$

Since $\mathcal{F}/(u,v)^{\#}$ is embeddable into a group $F/N_{uv^{-1}}$, and $N_{uv^{-1}}$ is the smallest normal subgroup with this property, it follows by [4], 12.3, that

$$N_{uv^{-1}} = gpn(st^{-1};\ (s,t) \in (u,v)^{\#}). \tag{2}$$

Hence by (1), (2): $(a,b)^{\#} \subseteq (u,v)^{\#}$ if and only if $N_{ab^{-1}} \subseteq N_{uv^{-1}}$, which gives the first statement of the Lemma. The second statement is known [11]. □

In terms of normal subgroups the above Questions are:

Question 1′ Does $N_{ab^{-1}} = V_{ab^{-1}}$ hold for each semigroup law $a = b$?

Question 2′ Does $V_{ab^{-1}} \subseteq V_{uv^{-1}}$ imply $N_{ab^{-1}} \subseteq N_{uv^{-1}}$ for semigroup laws $a = b$ and $u = v$?

We shall prove that for each semigroup law $a=b$ there exists a semigroup law $u = v$ such that the fully invariant closure of ab^{-1} coincides with the End^{+}-invariant normal closure of uv^{-1}. This will imply the equivalence of the Questions.

Theorem *For every n-variable semigroup law $a = b$ there exists an $n+1$-variable semigroup law $u=v$ such that $V_{ab^{-1}} = N_{uv^{-1}}$.*

Corollary *The Questions 1 and 2 are equivalent.*

Proof We have to show that for each semigroup law $a = b$ the equality holds: $N_{ab^{-1}} = V_{ab^{-1}}$. Take $u = v$ as in the Theorem, then $V_{ab^{-1}} \overset{\mathrm{T}}{=} N_{uv^{-1}}$. By taking the fully invariant closure we get $V_{ab^{-1}} = V_{uv^{-1}}$. If Question 2 has a positive answer then we have $N_{ab^{-1}} = N_{uv^{-1}} \overset{\mathrm{T}}{=} V_{ab^{-1}}$, as required. □

Lemmas and Proof of the Theorem

Lemma 2 *Let $A(x,y) = B(x,y)$ be a balanced and cancelled semigroup law and the first letter in $A(x,y)$ is x. Then there exist $a_i = a_i(x,y)$, $b_i = b_i(x,y) \in \mathcal{F}$, $i = 1,2$, such that*

$$(i)\ \ x^{-1}y = a_1 b_1^{-1} \cdot (A^{-1}B)^{b_1^{-1}}, \qquad (ii)\ \ xy^{-1} = a_2^{-1} b_2 \cdot (AB^{-1})^{\varepsilon b_2},\ \varepsilon = \pm 1,$$

$$(iii)\ \ F = \mathcal{F}\mathcal{F}^{-1} N_{AB^{-1}} = \mathcal{F}^{-1}\mathcal{F} N_{AB^{-1}}.$$

Proof Since the law $A=B$ is cancelled, it can be written as $x \cdot a_1 = y \cdot b_1$, which gives $A^{-1}B = a_1^{-1}x^{-1}yb_1$ and hence (i). The law $A=B$ (or $B=A$) can be written as $a_2 \cdot x = b_2 \cdot y$. In the first case $AB^{-1} = a_2xy^{-1}b_2$ gives $xy^{-1} = a_2^{-1}b_2 \cdot (AB^{-1})^{b_2}$. If $B = a_2 \cdot x,\ A = b_2 \cdot y$, then $xy^{-1} = a_2^{-1}b_2 \cdot (AB^{-1})^{-b_2}$, which gives (ii).

Since $(A^{-1}B) = (AB^{-1})^{B^{-1}} \in N_{AB^{-1}}$, we get from (i), that $x^{-1}y \in \mathcal{F}\mathcal{F}^{-1} \; mod\, N_{AB^{-1}}$, which holds under every substitution elements from $\mathcal{F}$ for x and y. Since every word in F is a product of words in $\mathcal{F} \cup \mathcal{F}^{-1}$, we get $F = \mathcal{F}\mathcal{F}^{-1}N_{AB^{-1}}$. Similarly, from (ii) we get $F = \mathcal{F}^{-1}\mathcal{F}N_{AB^{-1}}$. □

The following Lemma is well known in terms of a group of fractions and Ore conditions.

Lemma 3 *Let $a = b$ be a nontrivial semigroup law, and $g_1, g_2, \ldots, g_n$ be elements in F. Then there exist elements $s_1, s_2, \ldots, s_n$ and d in $\mathcal{F}$ such that $g_i = s_i d^{-1} \; mod\, N_{ab^{-1}}$.*

Proof By [6], the law $a = b$ implies balanced and cancelled binary law $A = B$. Since $N_{AB^{-1}} \subseteq N_{ab^{-1}}$, the inclusions in Lemma 2 are valid $mod\, N_{ab^{-1}}$. Then by (iii) we have *modulo* $N_{ab^{-1}}$: $g_i = a_i b_i^{-1}$ for some $a_i, b_i \in \mathcal{F}$. For $n = 2$, $g_1 = a_1 b_1^{-1}$, $g_2 = a_2 b_2^{-1}$. Also by (iii), there exist c, d such that $b_2^{-1} b_1 = cd^{-1}$. We introduce $r := b_1 d = b_2 c$, then $g_1 = a_1 b_1^{-1} = a_1 d d^{-1} b_1^{-1} = a_1 d r^{-1} =: sr^{-1}$, $g_2 = a_2 b_2^{-1} = a_2 c c^{-1} b_2^{-1} = a_2 c r^{-1} =: tr^{-1}$, $s, t, r \in \mathcal{F}$. So, by repeating this step we can write $g_1, \ldots, g_n$ with a "common denominator" $mod\, N_{ab^{-1}}$ as required. □

To compare End^{+}-invariant and End-invariant closures of words we make an observation that by positive endomorphisms we can map xy^{-1} into any word $g \in F \; mod \; N_{ab^{-1}}$ if write $g = st^{-1}$ and map x to s, and y to t.

Lemma 4 *There exists an automorphism $\alpha \in Aut\ F$ such that for any $w \in F$, N_{w^α} is fully invariant mod $N_{ab^{-1}}$, for any nontrivial $ab^{-1} \in \mathcal{F}\mathcal{F}^{-1}$. That is $V_w \subseteq N_{w^\alpha} N_{ab^{-1}}$.*

Proof Let $w = w(x_1, \ldots, x_n)$. We take $\alpha \in Aut\, F$ which maps $x_i \to x_i x_{n+1}^{-1}$, $i = 1, \ldots, n$ and leaves x_i, $i > n$, fixed. It is enough to show that for any $g_1, \ldots, g_n$ in F, $w(g_1, \ldots, g_n) \in N_{w^\alpha} N_{ab^{-1}}$. By Lemma 3, we write $g_i = s_i d^{-1} \; mod\, N_{ab^{-1}}$ and define $\nu \in End^{+}$ by $x_i^\nu = s_i$, $i \leq n$, and $x_{n+1}^\nu = d$. Then *modulo* $N_{ab^{-1}}$ we have $(x_i x_{n+1}^{-1})^\nu = g_i$ and $w(g_1, \ldots, g_n) =$ $w(x_1 x_{n+1}^{-1}, \ldots, x_n x_{n+1}^{-1})^\nu = (w(x_1, \ldots, x_n)^\alpha)^\nu \in N_{w^\alpha}^\nu \subseteq N_{w^\alpha}$, as required. □

Corollary 1 *For a nontrivial semigroup law $a = b$ the equality holds*

$$V_{ab^{-1}} = N_{(ab^{-1})^\alpha}.$$

Proof We have $ab^{-1} \in N^{\alpha^{-1}}_{(ab^{-1})^\alpha}$. Since α^{-1} is in End^+, then $N^{\alpha^{-1}}_{(ab^{-1})^\alpha} \subseteq N_{(ab^{-1})^\alpha}$ and hence $ab^{-1} \in N_{(ab^{-1})^\alpha}$, which gives

$$N_{ab^{-1}} \subseteq N_{(ab^{-1})^\alpha}. \qquad (3)$$

By Lemma 4 for $w := ab^{-1}$, by (3), and since $End^+ \subseteq End$, we have:

$$V_{ab^{-1}} \subseteq N_{(ab^{-1})^\alpha} N_{ab^{-1}} = N_{(ab^{-1})^\alpha} \subseteq V_{ab^{-1}},$$

which implies $V_{ab^{-1}} = N_{(ab^{-1})^\alpha}$.□

We denote by δ the endomorphism which maps $x_{n+1} \to 1$ and leaves other generators fixed, then $\delta \in End^+$. As above, $\alpha \in Aut\, F$ maps $x_i \to x_i x_{n+1}^{-1},\ i = 1, \ldots, n$ and leaves $x_i,\ i > n$, fixed.

Lemma 5 *Let $a = b$ be a nontrivial semigroup law, and $\mathcal{F}_{n+1}$ be a free subsemigroup generated by $x_1, ..., x_{n+1}$. Then for any positive word $p(x_1, ..., x_n)$, there exist positive words $u_i = u_i(x_1, ..., x_{n+1})$, $v_i = v_i(x_1, ..., x_{n+1})$, $i = 1, 2$, such that $p^\alpha = u_1 v_1^{-1} = u_2^{-1} v_2$ mod $(N_{ab^{-1}} \cap Ker\delta)$.*

Proof We show first that for any words $c, q \in \mathcal{F}_{n+1}$ the inclusion hold:

$$(*) \ \ c x_{n+1}^{-1} \in \mathcal{F}_{n+1}^{-1} \mathcal{F}_{n+1} \ mod \ (N_{ab^{-1}} \cap Ker\delta),$$

$$(**) \ \ x_{n+1}^{-1} q \in \mathcal{F}_{n+1} \mathcal{F}_{n+1}^{-1} \ mod \ (N_{ab^{-1}} \cap Ker\delta).$$

The law $a = b$ implies balanced and cancelled binary law $A = B$, so it is enough to prove the inclusions for the law $A(x, y) = B(x, y)$.

If apply δ to the balanced equality $A(c, x_{n+1}) = B(c, x_{n+1})$, it becomes trivial, and hence the word $AB^{-1}(c, x_{n+1})$ is in $Ker\,\delta$. Similarly we get $A^{-1}B(x_{n+1}, q) \in Ker\,\delta$. We put now c, x_{n+1} for x, y in (ii) (Lemma 2) to get $(*)$, and then put x_{n+1}, q in (i) (Lemma 2) to get $(**)$.

We continue the proof *modulo* $(N_{ab^{-1}} \cap Ker\delta)$. To show that: $p(x_1 x_{n+1}^{-1}, ..., x_n x_{n+1}^{-1}) \in \mathcal{F}_{n+1} \mathcal{F}_{n+1}^{-1}$, and $p(x_1 x_{n+1}^{-1}, ..., x_n x_{n+1}^{-1}) \in \mathcal{F}_{n+1}^{-1} \mathcal{F}_{n+1}$, we use induction on the length $|p| = m$. Let $p(x_1, ..., x_n) = c_m c_{m-1} ... c_2 c_1$, $c_i \in \{x_1, ..., x_n\}$, then $p^\alpha = c_m x_{n+1}^{-1} c_{m-1} x_{n+1}^{-1} ... c_2 x_{n+1}^{-1} c_1 x_{n+1}^{-1}$. For $m = 1$, $p^\alpha = c x_{n+1}^{-1} \in \mathcal{F}_{n+1} \mathcal{F}_{n+1}^{-1}$ and by $(*)$, $p^\alpha = c x_{n+1}^{-1} \in \mathcal{F}_{n+1}^{-1} \mathcal{F}_{n+1}$.

Let $|p| = m$, then $p = c_m c_{m-1} ... c_2 c_1$ and by inductive assumption $p^\alpha = c_m x_{n+1}^{-1} \cdot q r^{-1}$. Then by $(**)$, there exist $s, t \in \mathcal{F}_{n+1}$, such that $x_{n+1}^{-1} q = s t^{-1}$ and hence $p^\alpha = c_m (x_{n+1}^{-1} q) r^{-1} = c_m (s t^{-1}) r^{-1} = (c_m s)(rt)^{-1} \in \mathcal{F}_{n+1} \mathcal{F}_{n+1}^{-1}$.

Again for $|p|=m$, we get by assumption $p^\alpha = r^{-1}s\cdot c_1 x_{n+1}^{-1} = r^{-1}(sc_1)x_{n+1}^{-1}$. By $(*)$ for sc_1 instead of c, there exist $t, u \in \mathcal{F}_{n+1}$, such that $sc_1 x_{n+1}^{-1} = t^{-1}u$. Then $p^\alpha = r^{-1}(sc_1)x_{n+1}^{-1} = r^{-1}t^{-1}u = (tr)^{-1}u \in \mathcal{F}_{n+1}^{-1}\mathcal{F}_{n+1}$ as required. □

Proof of the Theorem

We have to show that for every nontrivial n-variable semigroup law $a = b$ there exists an $n+1$-variable semigroup law $u = v$ such that $V_{ab^{-1}} = N_{uv^{-1}}$.

By Lemma 5 for the words $a = a(x_1, ..., x_n)$ and $b = b(x_1, ..., x_n)$ we get respectively: $a^\alpha = u_1 v_1^{-1}\ mod\ (N_{ab^{-1}} \cap Ker\delta)$, and $b^\alpha = u_2^{-1} v_2\ mod\ (N_{ab^{-1}} \cap Ker\delta)$. Then $(ab^{-1})^\alpha = u_1 v_1^{-1} v_2^{-1} u_2 = u_2^{-1}(u_2 u_1)(v_2 v_1)^{-1} u_2\ mod\ (N_{ab^{-1}} \cap Ker\delta)$. We denote $u := u_2 u_1$, $v := v_2 v_1$, then

$$(ab^{-1})^\alpha = (uv^{-1})^{u_2}\ mod\ (N_{ab^{-1}} \cap Ker\delta) \tag{4}$$

This implies:

$$N_{(ab^{-1})^\alpha} \subseteq N_{uv^{-1}} N_{ab^{-1}} \tag{5}$$

and

$$N_{uv^{-1}} \subseteq N_{(ab^{-1})^\alpha} N_{ab^{-1}}. \tag{6}$$

To prove the equality

$$N_{(ab^{-1})^\alpha} = N_{uv^{-1}}, \tag{7}$$

we apply δ to (4). Since $\alpha\delta$ is the identity map on $x_i,\ i \leq n$, and δ is in End^+, we have that $ab^{-1} = (ab^{-1})^{\alpha\delta}$ is conjugate to $(uv^{-1})^\delta \in N_{uv^{-1}}^\delta \subseteq N_{uv^{-1}}$. This implies $N_{ab^{-1}} \subseteq N_{uv^{-1}}$ which, together with (5) gives $N_{(ab^{-1})^\alpha} \subseteq N_{uv^{-1}}$. Since by (3), $N_{ab^{-1}} \subseteq N_{(ab^{-1})^\alpha}$, it follows from (6), that $N_{uv^{-1}} \subseteq N_{(ab^{-1})^\alpha}$, and hence (7) holds.

Now, since by Corollary 1, $V_{ab^{-1}} = N_{(ab^{-1})^\alpha}$, we have by (7), the required equality $V_{ab^{-1}} = N_{uv^{-1}}$.□

Example of implications in semigroups [8]

The law $(xy)^2 = (yx)^2$ implies $xy^2 = y^2x$ for groups because we can apply the automorphism $\alpha : x \to x,\ y \to x^{-1}y$. For semigroups we can not use this automorphism. To prove that $(xy)^2 = (yx)^2$ implies $xy^2 = y^2x$ for semigroups we show first that $(xy)^2 = (yx)^2$ implies:

(i) $(yx)^2y = y(yx)^2$, (use the word $y(xy)^2$),
(ii) $x((yx)^2y)^2 = ((yx)^2y)^2x$, (use $(i)^\alpha$, $x^\alpha = xyx^2$, $y^\alpha = y$),
(iii) $((yx)^2y)^2 = (yx)^4y^2$, (use $((yx)^2y)((xy)^2y)$),
(iv) $(yx)^4 = (xy)^4$.

Then for some word p we start with $p \cdot xy^2$ and by using (i) - (iv) obtain $p \cdot y^2x$, which by cancellation, implies required $xy^2 = y^2x$.

Namely, for $p = (xy)^4$ we have

$$pxy^2 = (xy)^4xy^2 = x(yx)^2\underline{(yx)^2yy} \overset{(i)}{=} x(yx)^2\underline{y(yx)^2}y =$$

$$x((yx)^2y)^2 \overset{(ii)}{=} ((yx)^2y)^2x \overset{(iii)}{=} (yx)^4y^2x \overset{(iv)}{=} (xy)^4y^2x = py^2x,$$

which gives $pxy^2 = py^2x$ and hence $xy^2 = y^2x$ as required.

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