GROUPLAND

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Abstract

Groupland is a flat planet where groups live in areas defined by their properties, e.g. residually finite groups, groups of different growth types etc. This visualizes relations of properties, and formulation of problems as, for example, where do Engel groups live? We introduce the area where some open problems can be solved.

A law $u(x_1,...,x_n) = v(x_1,...,x_n)$ is called positive if u, v are written without inverses of variables. The simplest examples are $xy = yx$ and $x^{n+1} = x,\ n > 0$. If a group G satisfies a positive law, then the variety $var(G)$ has a basis of positive laws [9]. We note that groups satisfying positive laws can not contain a free non-abelian subsemigroup, denoted by $\mathcal{F}$.

In Fig.1 of Groupland we introduce the areas: of groups which have no laws, groups with only non-positive laws, and groups satisfying positive laws.

All groups containing a free non-abelian subsemigroup $\mathcal{F}$ are in the right half of Groupland. Groups in the left half do not contain $\mathcal{F}$.

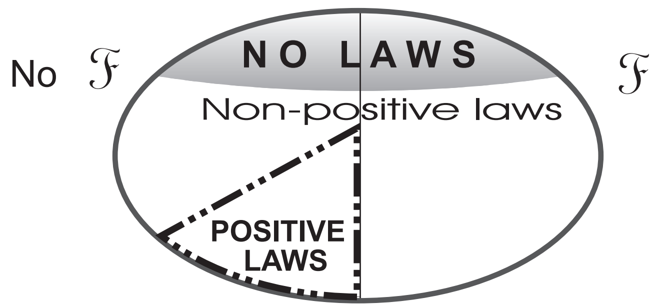

Fig.1, Groupland

We need the following results.

Lemma 1

(a) *If a group G is nilpotent-by-(finite exponent), then G satisfies a positive law.*
(b) *If a finitely generated group G is soluble-by-finite, then either G is nilpotent-by-finite or G contains $\mathcal{F}$.*
(c) *If G is a finitely generated periodic residually finite group which satisfies a nontrivial law, then G is finite.*

Proof (a) It was shown independently in [13] and [16], that nilpotent groups satisfy positive laws. So if a variety $\mathfrak{N}_c$ of c-nilpotent groups satisfies a positive law $u(x_1,...,x_c)=v(x_1,...,x_c)$, then a nilpotent-by-Burnside variety $\mathfrak{N}_c\mathcal{B}_e$ satisfies the positive law $u(x_1^e,...,x_c^e)=v(x_1^e,...,x_c^e)$.

(b) It is shown in [19] 4.7, that a finitely generated soluble group with no free subsemigroup on two generators must be polycyclic. Also by [19] 4.12, if G is a polycyclic group, then G either has a nilpotent subgroup of finite index or G contains a free subsemigroup on two generators. These imply (b).
(c) This result can be found in [27] IV.2.6. □

Since 1953 it was conjectured that a group satisfies positive laws if and only if it is an extension of a nilpotent group by a group of finite exponent. The counterexample was found only in 1996 [17], where it was shown that there exist groups satisfying positive laws, which are not even (locally soluble)-by-(finite exponent). By [3], relatively free groups of prime exponent ≥ 665 and of finite rank > 1 have exponential growth. In Fig.2 of Groupland these groups are in the area with the small star.

We add now four more areas to the picture.

Statement 1 *Fig.2 of Groupland shows the areas of:*
(i) *groups of polynomial growth (hatched pattern),*
(ii) *groups of intermediate growth (a small white circle),*
(iii) *polycyclic groups (a small inner grey ellipse),*
(iv) *residually finite groups (a large dotted ellipse).*

Proof (i) By results of Gromov [5], Milnor [14] and Wolf [26], a group G has polynomial growth if and only if it is nilpotent-by-finite. So by Lemma 1 (a), groups of polynomial growth satisfy positive laws.
(ii) All known finitely generated infinite torsion groups of exponent zero, in particular the groups of intermediate growth, are residually finite and hence by Lemma 1 (c), they do not satisfy any law (see e.g. [3], ([15], 34.63), [4], [6], [23]).
(iii) The area of polycyclic groups is defined by Lemma 1 (b). As an example of a polycyclic group containing $\mathcal{F}$ we have a group $G = \langle a, b, c \mid [a, b] = 1,\ a^c = ab,\ b^c = a^2 b\rangle$, which by ([15] 32.35) generates whole metabelian variety $\mathfrak{A}^2$. So G is not nilpotent-by-finite and hence by Lemma 1 (b), it contains $\mathcal{F}$.
(iv) By P. Hall, nilpotent-by-finite groups are residually finite [7], and polycyclic groups are also residually finite ([15] 32.1). □

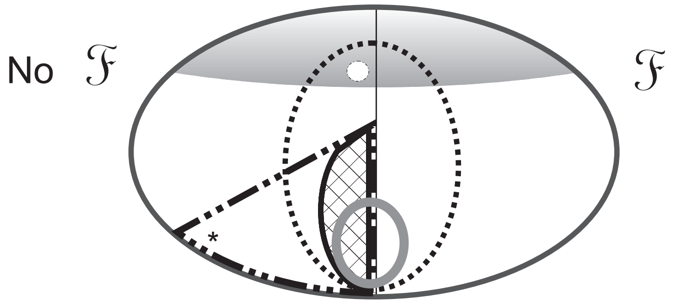

Fig.2, Groupland

We denote by $\mathfrak{B}_e$ the Restricted Burnside variety of exponent e, i.e. the variety generated by all finite groups of exponent e. All groups in $\mathfrak{B}_e$ are locally finite of exponent e. The existence of such varieties for each positive integer e follows from the positive solution of the Restricted Burnside Problem and relies on classification of finite simple groups (see [25]). Clearly $\mathfrak{S}_n\mathfrak{B}_e \subseteq \mathfrak{S}_n\mathcal{B}_e$.

Definition [1] We define an *SB-group* to be one lying in some product of finitely many varieties each of which is either soluble or a $\mathfrak{B}_e$ (for varying e).

In particular the class of SB-groups contains all groups G which are soluble-by-(locally finite of finite exponent), that is $G \in \mathfrak{S}_n\mathfrak{B}_e$ for some n, e.

Statement 2 *Fig.3 of Groupland contains new areas of:*
(i) SB-groups (inner horizontal ellipse),
(ii) groups which are nilpotent-by-(locally finite of finite exponent), $G \in \mathfrak{N}_c\mathfrak{B}_k$ *for some* c, k *(lined and hatched areas).*

Proof (i) As an example of an SB-group containing $\mathcal{F}$, we have a free metabelian group of rank > 1 [13]. SB-groups without $\mathcal{F}$ may satisfy positive laws as, for example, nilpotent-by-finite groups. However a cartesian product of metabelian nilpotent groups of growing nilpotency class does not satisfy positive laws.
(ii) It follows from [1] Thm.B, that an SB-group G satisfies a positive law if and only if G is nilpotent-by-(locally finite of finite exponent). So the intersection of areas of SB-groups and groups satisfying positive laws coincides with the area of groups which are nilpotent-by-(locally finite of finite exponent). □

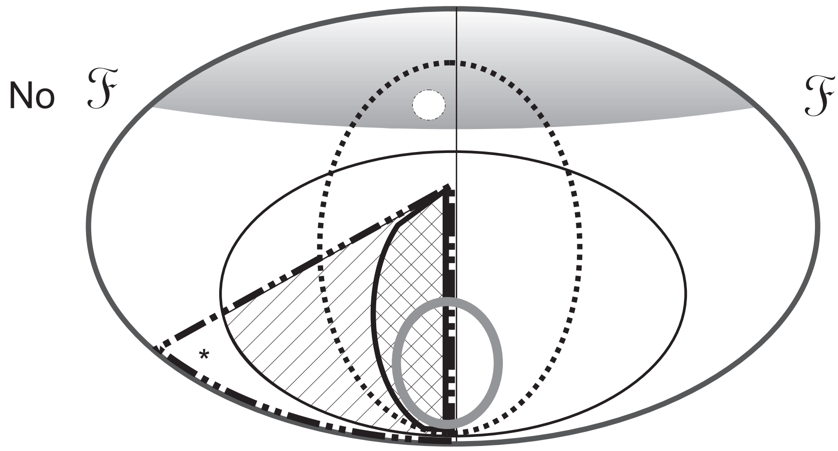

Fig.3, Groupland

Class $\mathcal{C}$. The large class $\mathcal{C}$, introduced in [1] is a sum $\mathcal{C} = \cup\Delta_n$, where Δ_1 is the class of SB-groups. $\Delta_{n+1} = \{\text{groups, locally in } \Delta_n\} \cup \{\text{groups, residually in } \Delta_n\}$.

The class $\mathcal{C}$ consists of groups accessible to analysis by methods of group theory used in the textbooks. The groups outside the class $\mathcal{C}$, one may conjecture, require special techniques of Adian-Novikov and Ol'shanskii. It is possible that many Questions have negative answers in general, and have affirmative answers in the class $\mathcal{C}$. We demonstrate some of them.

Statement 3 *Fig.4 of Groupland shows the area of the class* $\mathcal{C}$ *(the area inside the dashed ellipse, including the solid ellips.*

Proof By definition all SB-groups, all residually finite and residually soluble groups are in $\mathcal{C}$. It is shown in [1] Thm.B, that a group G in the class $\mathcal{C}$ satisfies a positive law if and only if G is nilpotent-by-(locally finite of finite exponent).□

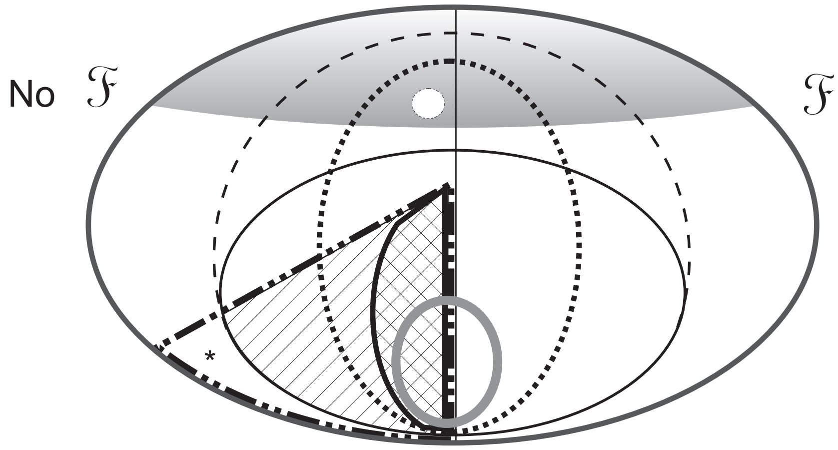

Fig.4, Groupland

We formulate now several known questions which are open in general but have affirmative answers in the class $\mathcal{C}$.

The question **whether every n-Engel group is locally nilpotent** is known since 1936, when Zorn proved that a finite n-Engel group is nilpotent. Positive answers are known for soluble Engel groups (Gruenberg, 1953), residually finite Engel groups (Wilson, 1991), profinite Engel groups (Wilson and Zelmanov, 1992), see [2]. The problem is still open in general.

The question **whether n-Engel group satisfies a positive law** was posed by Shirshov in 1963 [22]. He proved that 2-Engel law $[[x,y],y]$ is equivalent to $xy^2x = yx^2y$, and 3-Engel law is equivalent to two positive laws. Then in [10] Maj et all. proved that torsion-free 4-Engel group satisfies positive laws. In 1998 Traustason [24] proved that every 4-Engel group satisfies positive laws. For $n > 4$ the problem is open.
However it was proved in 1998 by Burns and Medvedev [2], that in the class $\mathcal{C}$ every n-Engel group is contained in a variety $\mathfrak{N}_{c(n)}B_{e(n)} \cap \mathfrak{B}_{e(n)}\mathfrak{N}_{c(n)}$, which gives affirmative answer for both questions in the class $\mathcal{C}$.

In [20], [21] authors defined collapsing group as one for which there exists n such that for every n-element subset $S \subseteq G$, the inequality $|S^n| < n^n$ holds. A question **whether collapsing group satisfies a positive law** was answered affirmatively for residually finite and soluble collapsing groups. The problem is open in general, however it was shown in [11], that in the class $\mathcal{C}$ every collapsing group satisfies a positive law.

Let G be a finitely generated relatively free group. A conjecture, that G' **is finitely generated if and only if G satisfies a positive law**, also open in general, has affirmative answer in the class $\mathcal{C}$ [12].

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