Collapsing groups and positive laws

Olga Macedońska

September 11, 2013

Abstract

The paper concerns the question of A. Shalev: is it true that every collapsing group satisfies a positive law? We give a positive answer for groups in a large class $\mathcal{C}$, including all soluble and residually finite groups.

Let $u(x, y), v(x, y)$ be some words in a free canellation semigroup $\mathcal{F}_2$, generated by x, y. We say that elements g, h in a group G satisfy a positive relation $u(x, y) = v(x, y)$ if the equality $u(g, h) = v(g, h)$ holds. A group G satisfies a binary positive law $u(x, y) = v(x, y)$ if every pair of elements in G satisfies the relation $u(x, y) = v(x, y)$. We recall that every n-variable positive law implies a binary positive law [7].

We say that the relation $u(x, y) = v(x, y)$ is of degree n, if it is cancelled (the first (and the last) letters in u and v are different), balanced (the exponent sum of x (and of y) is the same in u and v) and the length of u (equal to the length of v) is n.

In a group G without a free nonabelian subsemigroup any two elements satisfy some positive relation. If all these relations have a restricted degree $\leq n$, then G is called n-collapsing group (cf. [13]).

There is an inclusion for classes of groups with the following properties: satisfying positive laws, collapsing, and groups without free nonabelian subsemigroups.

$$\{\text{positive law}\} \subseteq \{\text{collapsing}\} \subset \{\text{without } \mathcal{F}_2\}. \tag{1}$$

1991 AMS Subject Classification 20F19

The second inclusion in (1) is strict. Indeed, if G is the direct product of nilpotent groups of classes $i = 1, 2, 3, ...$, then G has no free subsemigroup, but is not collapsing, because the degree of relations depends on the class of nilpotency [9]. Finitely generated examples give the Shmidt group by Ol'shanskii [10], and the infinite torsion groups without laws [3], [4], because collapsing groups satisfy some commutator law [14].

It is an open problem: whether the first inclusion in (1) is strict. The question was posed by A.Shalev in [14] as:

Question Is it true that every collapsing group satisfies a positive law?

For residually finite groups the positive answer was given in [14]. Our main result answers the question affirmatively for groups in a large class $\mathcal{C}$, including soluble and residually finite groups. The class $\mathcal{C}$ was introduced in [2].

It was known since 1953 [9], that groups, which are nilpotent-by-finite exponent, satisfy a positive law. Till 1996 all known examples of groups satisfying positive laws were nilpotent-by-finite exponent.

We recall the known inclusions for smaller classes of groups:

$$\left\{\begin{array}{c}\text{nilpotent-by-}\\ \text{locally finite of}\\ \text{finite exponent}\end{array}\right\} \subset \left\{\begin{array}{c}\text{nilpotent-}\\ \text{by-finite}\\ \text{exponent}\end{array}\right\} \subset \left\{\begin{array}{c}\text{positive}\\ \text{law}\end{array}\right\} \subseteq \{\text{collapsing}\}$$

The first inclusion is strict because the groups F/F^n for n odd, ≥ 665 are not locally finite [1]. The second inclusion is also strict because of the group of Ol'shanskii and Storozhev [11].

In [2] we introduced the large class $\mathcal{C}$, where every group of a finite exponent is locally finite.

To recall the definition we denote by $\mathfrak{B}_e$ so called restricted Burnside variety of exponent e, i.e. the variety generated by all finite groups of exponent e. All groups in $\mathfrak{B}_e$ are locally finite of exponent e. The existence of such varieties for each positive integer e follows from the positive solution of the Restricted Burnside Problem (Kostrikin [6], Zelmanov [15], [16]).

We define an *SB-group* to be one lying in some product of finitely many varieties each of which is either soluble or a $\mathfrak{B}_e$ (for varying e). It follows from the definition, that the class of SB-groups, is closed for extensions.

The class $\mathcal{C}$ is obtained from the class of all SB-groups by repeated applications of the operations L, R and E, where for any group-theoretic class $\mathcal{X}$ of groups (see [12]), $L\mathcal{X}$ denotes the class of all groups locally in $\mathcal{X}$, $R\mathcal{X}$ the class of groups residually in $\mathcal{X}$ and $E\mathcal{X}$ the class of extensions of groups in $\mathcal{X}$ by groups in $\mathcal{X}$. In particular residually finite and residually soluble groups are in $\mathcal{C}$. Every group of a finite exponent in $\mathcal{C}$ is locally finite. The class $\mathcal{C}$ contains all soluble varieties, all restricted Burnside varieties and the semigroup of varieties they generate.

Note: The class $\mathcal{C}$ is obtained from the class of all finite and soluble groups by repeated applications of the operations L, R and E. In [2], in the definition of the class $\mathcal{C}$ the operator E is missing. All results are valid for the extended definition.

In [2] we proved that the class $\mathcal{C}$ cuts out the nilpotent - by - locally finite of finite exponent groups from the class of groups with positive laws:

$$\left\{ \begin{array}{c} \text{nilpotent-by-locally finite} \\ \text{of finite exponent} \end{array} \right\} = \left\{ \begin{array}{c} \text{positive} \\ \text{law} \end{array} \right\} \cap \text{ C.}$$

Our result in this paper says that every collapsing group in the class $\mathcal{C}$ is nilpotent-by-locally finite of finite exponent, that is

$$\left\{ \begin{array}{c} \text{nilpotent-by-locally finite} \\ \text{of finite exponent} \end{array} \right\} = \{\text{collapsing}\} \cap \text{ C.}$$

Our proof is based on the two known Theorems.

Theorem 1 *(cf. Theorem B, [14]) There exist functions f, g such that any finite n-collapsing group G has a normal subgroup N such that $exp(G/N)$ divides $f(n)$ and every 2-generator subgroup of N is nilpotent of a class at most $g(n)$.*

Theorem 2 *[2] If a group G in the class $\mathcal{C}$ satisfies a positive law of degree k, then G is an extension of a nilpotent group of class $\leq c'(k)$ by a locally finite group of exponent dividing $e'(k)$:*

$$G \in \mathfrak{N}_{c'(k)}\mathfrak{B}_{e'(k)},$$

where the integers $c'(k)$, $e'(k)$ depend on k only.

The following Lemma extends Theorem A [14].

Lemma 1 *If G is any* **residually finite** *n-collapsing group then there exist functions c and e such that*

$$G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)},$$

where the integers $c(n)$, $e(n)$ depend on n only.

Proof Since G is residually finite there is a chain

$$G \geq N_1 \geq N_2 \geq \dots$$

of normal subgroups of G such that $|G : N_i| < \infty$ and $\cap_i N_i = \{1\}$.

Since G/N_i is finite n-collapsing, then by Theorem 1, it contains a normal subgroup, every 2-generator subgroup of which is nilpotent of a class at most $g(n)$. Then by A. Malcev [9], this normal subgroup satisfies a positive law $P_g(x, y) = Q_g(x, y)$. Again by Theorem 1, the quotient has exponent dividing $f(n)$, which implies that G/N_i satisfies the binary positive law $P_g(x^f, y^f) = Q_g(x^f, y^f)$ of a degree $k = k(n)$, say, which depends on n only. Since G is a subcartesian product of the G/N_i, it satisfies the same law.

Now by Theorem 2 there exist functions c' and e' such that the residually finite group G satisfying a positive law of degree k belongs to $\mathfrak{N}_{c'(k)}\mathfrak{B}_{e'(k)}$. Since k is a function of n only, we put $c(n) = c'(k)$ and $e(n) = e'(k)$, which finishes the proof.

Lemma 2 *Any n-collapsing group G in a product $\mathfrak{B}_{e_1}\mathfrak{S}_d$ of a restricted Burnside variety and a soluble variety satisfies*

$$G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)},$$

for $c(n)$, $e(n)$ as in Lemma 1.

Proof Let H be a finitely generated subgroup in the group G. As a collapsing group, H does not contain a free non-abelian subsemigroup, and by [8, Corollary 3], all its derived subgroups are finitely generated. Since by assumption $H^{(d)}$ is in $\mathfrak{B}_{e_1}$, it is finite. Let Z denotes the centralizer of $H^{(d)}$ in H, which then has a finite index in H. Then Z is finitely generated (because H is finitely generated). Moreover, Z is soluble, because $1 = [H^{(d)}, Z] \supseteq [Z^{(d)}, Z] \supseteq Z^{(d+1)}$.

The finitely generated soluble group Z without free non-abelian subsemigroups is, by [12, Theorems 4.7, 4.12], nilpotent-by-finite and hence residually finite. So H, as a finite extension of Z, is residually finite and by Lemma 1 $H \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$. Since the same is true for every finitely generated subgroup H in G, we obtain $G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$, as required.

Theorem 3 *Collapsing groups in the class $\mathcal{C}$ are nilpotent-by-locally finite of finite exponent and hence satisfy a positive law.*

Proof We show first that every n-collapsing SB-group belongs to $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$ for $c(n)$, $e(n)$ as in Lemma 1.

Let G be an n-collapsing SB-group, i.e. $G \in \mathfrak{V}_1\mathfrak{V}_2 \dots \mathfrak{V}_t$, where each variety $\mathfrak{V}_i$ is either soluble or a $\mathfrak{B}_e$ for some e. The product of varieties is associative. By Lemma 2, we exchange (starting from the right) every pair of the type $\mathfrak{B}\mathfrak{S}$ for some pair of the type $\mathfrak{N}\mathfrak{B}$, and obtain that G belongs to a soluble-by-restricted Burnside variety. We shall see that G is residually finite. Let H be a finitely generated subgroup in the n-collapsing group $G \in \mathfrak{S}_{c_1}\mathfrak{B}_{e_1}$. Then H is a finite extension of a soluble normal subgroup N, say. Being soluble without free non-abelian subsemigroups, N is then locally: nilpotent-by-finite [12], and hence residually finite. So H, as a finite extension of N, is residually finite and by Lemma 1, $H \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$. Since the same is true for every finitely generated subgroup H in G, we obtain $G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$.

Let now G be an n-collapsing group in the class $\mathcal{C}$. The dependence of the above parameters $c(n)$ and $e(n)$ on n only, implies that if in the group G each finitely generated subgroup is in $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$, then $G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$. Similarly, if G is a subcartesian product of n-collapsing groups in $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$, then again $G \in \mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$. Finaly, if an n-collapsing group G is an extension of a group in $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$ by another group in $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$, then G is an SB-group and hence is in $\mathfrak{N}_{c(n)}\mathfrak{B}_{e(n)}$, which finishes the proof.

References

[1] S.I.Adian, The problem of Burnside and identities in groups, *Nauka, Moscow, 1975.* (Russian) (see also, trans. J.Lennox and J.Wiegold, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **92***, Springer-Verlag, Berlin, 1979*)

[2] Robert G.Burns, Olga Macedońska, Yuri Medvedev, Groups Satisfying Semigroup Laws, and Nilpotent-by-Burnside Varieties, *Journal of Algebra* **195** *(1997), 510-525.*

[3] R.I.Grigorchuk, On the growth degrees of p-groups and torsion-free groups, *Math. Sb.* **126** *(1985), 194-214.*

[4] N.Gupta and S.Sidki, Some infinite p-groups, *Algebra i Logika* **22** *(1983), 584-589.*

[5] P.Hall, On the finiteness of certain soluble groups, *Proc. London Math. Soc.* **9***(1959), 595-622.*

[6] A.I.Kostrikin, On Burnside problem, *Izv. AN SSSR* **23***, (1959), 3-34.*

[7] Jacques Lewin and Tekla Lewin, Semigroup laws in varieties of soluble groups, *Proc. Camb. Phil. Soc.* **65** *(1969),1-9.*

[8] P.Longobardi, M.Maj and A.H.Rhemtulla, Groups with no free subsemigroups, *Trans.Amer.Math.Soc.,* **347***, 4, (1995), 1419-1427).*

[9] A.I.Mal'cev, Nilpotent semigroups, *Uchen. Zap. Ivanovsk. Ped. Inst.* **4** *(1953), 107-111.*

[10] A.Yu.Ol'shanskii, An infinite group with subgroups of finite orders, *Izv. AN SSSR, Mat,* **44***, 2, (1980), 309-321.*

[11] A.Yu.Ol'shanskii and A.Storozhev, A group variety defined by a semigroup law, *J. Austral. Math. Soc. (Series A),* **60***, (1996), 255-259.*

[12] J.M.Rosenblatt, Invariant measures and growth conditions, *Trans. Am. Math. Soc.* **193***, (1974), 33-53.*

[13] J.F.Semple and A.Shalev, Combinatorial conditions in residually finite groups, I, *J.Algebra* **157** *(1993), 43-50.*

[14] A.Shalev, Combinatorial conditions in residually finite groups, II, *J.Algebra* **157** *(1993), 51-62.*

[15] E.I.Zelmanov, The solution of the restricted Burnside problem for groups of odd exponent, *Math. USSR-Izv.* **36** *(1991), 41-60.*

[16] E.I.Zelmanov, The solution of the restricted Burnside problem for 2-groups, *Mat. Sb.* **182** *(1991), 568-592.*

Institute of Mathematics,
Silesian Technical University,
ul. Kaszubska 23,
44-100 Gliwice, Poland
e-mail: olga@zeus.polsl.gliwice.pl