Positively discriminating groups

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Abstract

A group is positively discriminating if any finite subset of positive equations $u=v$, which are not laws in G, can be simultaneously falsified in G. All known groups, which are not positively discriminating, satisfy positive laws. The problem whether every group without positive laws must be positively discriminating is open. We give an affirmative answer to the problem in the class of locally graded groups.

An equation in a group is an expression of the form $u = v$, where $u=u(x_1, ..., x_n)$, $v=v(x_1, ..., x_n)$ are different words (v may be the empty word 1) in the free group F, freely generated by $x_1, x_2, ...$. If $n = 2$, the equation is called *binary* and we use x, y instead of x_1, x_2. The equation is called *positive* if u and v are written *without inverses* of x_i's. A positive equation is called *balanced* if the exponent sum of x_i is the same in u and v for each fixed i. A balanced equation $u = v$ is of *degree n* if the x-length of u and v is equal n. We say that the n-tuple of elements $g_1, ..., g_n$ in G satisfies the equation $u=v$, if under substitution $x_i \to g_i$ we get the equality $u(g_1, ..., g_n) = v(g_1, ..., g_n)$. If the equality does not hold, the n-tuple *falsifies* the equations. The equation $u=v$ is *the law* in a group G if every n-tuple of elements in G satisfies this equation. The equation is *the non-law* in G, if it is not the law in G, hence there is an n-tuple, which falsifies the equation.

Let $\mathfrak{V}$ be a finite set of equations. Since the equations need not be cancelled, we can assume that for some n all the equations in $\mathfrak{V}$ are written on n variables. If there is an n-tuple in a group G, which falsifies each equation in $\mathfrak{V}$, we say that $\mathfrak{V}$ can be *simultaneously* falsified in the group G. For example, in symmetric group S_3 the set of two equations $\{xy = y,\ xy^2 = x\}$ can be simultaneously falsified by pair of elements $a, d \in S_3$, of orders $2, 3$, respectively: $ad \neq d,\ ad^2 \neq a$. However the set $\{xy^2 = x,\ xy^3 = x\}$ can not be simultaneously falsified, because either y^2 or y^3 has the image 1 in S_3 and hence each pair satisfies at least one equation.

More examples The following binary sets $\mathfrak{V}$ of non-laws can not be simultaneously falsified. Each pair of elements satisfies some equation in $\mathfrak{V}$:

1. Quaternion group Q_8 : $\mathfrak{V}=\{\,xy=yx,\ x^3y=yx\,\}$.
2. Cyclic group C_3 : $\mathfrak{V}=\{\,xy=1,\ x^2y=1,\ xy=x,\ xy=y\,\}$.

We recall that a group G is *discriminating* (see [10] 17.12, 17.23), if any finite subset $\mathfrak{V}$ of non-laws in G, can be simultaneously falsified in G. In terms of [1] it means that G discriminates the free group in *var* G. If we consider only the subsets $\mathfrak{V}$ of positive equations, or of binary equations, we can speak of positively or binary discriminating groups, respectively.

Definition A group G is called positively discriminating if for any finite subset $\mathfrak{V}$ of positive equations $u = v$, which are non-laws in G, there exist elements $g_1, ..., g_n$ in G, such that $u(g_1, ..., g_n) \neq v(g_1, ..., g_n)$ for all $u = v$ in $\mathfrak{V}$ *simultaneously.*

The consequence of the definition is the following

Proposition 1 If G contains a free non-cyclic subsemigroup, then G is positively discriminating. Every discriminating group is positively discriminating, however the converse is not true.

Proof The first statement follows because each free non-cyclic subsemigroup contains the free subsemigroup of infinite rank, where every subset of positive equations can be falsified simultaneously.

By definition, every discriminating group is positively discriminating. The converse is not true. Take the group $G = A_5 \times F/F''$, where A_5 is the alternating group and F/F'' – a free metabelian group of rank > 1. The group G is positively discriminating, because by [9], F/F'' contains a free non-cyclic subsemigroup. However G is not discriminating, because the set of the commutator non-laws $\mathfrak{V} = \{ \, [\,[x^d, y], [x, y]\,] = 1\,; \; d|60 \, \}$, can not be simultaneously falsified (every pair of elements in G satisfies at least one of them). $\square$

Proposition 2 (cf [10] 17.32) No finite group is positively discriminating.

Proof If $|G| = n$, we take $\mathfrak{V}$ to consist of $n(n + 1)$ nontrivial equations $x_i = x_j$, $i, j = 1, 2, ..., n+1$. Since there is more variables then elements in G, the pigeon-hole principle implies that the equations in $\mathfrak{V}$ can not be falsified in G simultaneously. $\square$

All known groups, which are not positively discriminating, e.g. finite groups, satisfy positive laws, and the groups without positive laws, e.g. free soluble groups ([10] 32.23), [9], are positively discriminating. So the natural question arises:

Question *Must a group without positive laws be positively discriminating?*

We give an affirmative answer in a large class of groups. First we note that in the class of groups which do not satisfy positive laws, the definition of positively discriminating group can be restricted to only binary equations. Such a group is positively discriminating if and only if it is binary positively discriminating.

Theorem 1 A group G, which does not satisfy positive laws, is positively discriminating if and only if for any finite subset $\mathfrak{V}$ of positive ***binary*** equations $u(x, y) = v(x, y)$, there exist elements g, h in G, such that $u(g, h) \neq v(g, h)$ for all equations in $\mathfrak{V}$ simultaneously.

Proof The "only if" part is clear, because if any finite subset of positive equations can be simultaneously falsified in G, then the same is true for any finite subset of binary positive equations.

Conversely, let G be a binary positively discriminating group, so any finite subset of *binary* positive equations *can be* simultaneously falsified in G. Assume that G is not positively discriminating, then there exists a finite subset $\mathfrak{V}$ of positive equations on $n > 2$ variables, which can not be simultaneously falsified in G. Let α maps $x_i \to xy^i$, then the subset $\mathfrak{V}$ defines the subset $\mathfrak{V}^\alpha$ of non-trivial binary positive equations.

Since $\mathfrak{V}$ can not be simultaneously falsified, every n-tuple of elements in G satisfies at least one equation in $\mathfrak{V}$. So for every pair $g, h \in G$, the n-tuple $gh, gh^2, ..., gh^n$ satisfies some equation $u(x_1, ..., x_n) = v(x_1, ..., x_n)$ in $\mathfrak{V}$, that is the equality holds $u(gh, gh^2..., gh^n) = v(gh, gh^2..., gh^n)$. It means that the pair g, h satisfy the binary equation $u(xy, xy^2..., xy^n) = v(xy, xy^2..., xy^n)$ in $\mathfrak{V}^\alpha$. So the set $\mathfrak{V}^\alpha$ of binary positive equations can not be simultaneously falsified in G, which is the contradiction. $\square$

We need the following two technical lemmas.

Lemma 1 Let G be finitely generated and does not contain free non-cyclic subsemigroups. If G/N is nilpotent-by-finite, then N is finitely generated.

Proof By assumption, G/N contains a nilpotent normal subgroup H/N of finite index, hence H and H/N are finitely generated. Then by ([10] 31.12), there is a finite normal series with cyclic factors $H = N_0 \triangleright N_1 \triangleright ... \triangleright N_m = N$. We know that N_0 is finitely generated and assume, that N_i is finitely generated. Since by assumption G does not contain free non-cyclic subsemigroups, and the group N_i/N_{i+1} is cyclic, it follows from ([5], Lemmas 5 and 1) that N_{i+1} is finitely generated, which accomplishes the induction, and proves that N is finitely generated. $\square$

Lemma 2 A finitely generated group, which is finite-by-nilpotent-by-finite, is nilpotent-by-finite.

Proof It suffices to show that a finite-by-nilpotent group is nilpotent-by-finite. Let G be a finitely generated group and let N be its finite normal subgroup such that G/N is nilpotent of class c. Then $\gamma_{c+1}(G) \subseteq N$. Moreover, since G is finitely generated and N is finite, then the centralizer C of N in G is a normal subgroup of finite index in G. Hence $\gamma_{c+2}(C) = [\gamma_{c+1}(C), C] \subseteq [N, C] = 1$, so C is nilpotent normal subgroup of finite index in G, which means that G is nilpotent-by-finite as required. $\square$

We show that the Question, whether a group without positive laws must be positively discriminating, has an affirmative answer in the large class of so called locally graded groups, introduced in 1970 by Černikov. This class was defined to avoid groups with finitely generated infinite simple sections, such as infinite Burnside groups and Ol'shanskii-Tarski monster.

A group G is called *locally graded* if every nontrivial finitely generated subgroup in G has a proper subgroup of finite index.

Theorem 2 Every locally graded group without positive laws is positively discriminating.

Proof In view of Proposition 1, it suffices to consider only locally graded groups without free non-cyclic subsemigroups, which do not satisfy positive laws. Let G be such a group. We show that the assumption that G is not positively discriminating leads to a contradiction.

If G is not positively discriminating then by Theorem 1, there is a finite subset $\mathfrak{V}$ of binary positive equations, such that every pair $g, h \in G$ satisfy some of these equations. We can assume, that $\mathfrak{V}$ consists of balanced binary positive equations

$u_i(x,y)=v_i(x,y)$, because if elements satisfy an equation $u=v$, then they satisfy the balanced equation $uv=vu$. If elements satisfy an equation $u=1$, then they also satisfy the balanced equation $ux=xu$. Since the equations need not be cancelled, we can assume all of them of the same degree n, say.

So for any two-element set $S=\{g,h\}$ in G, some words $u(g,h)$ and $v(g,h)$ in S^n are equal. Hence we have $|S^n|<2^n$ which means that G is the $(n,2)$-collapsing group [11]. Now there are two possibilities. If G is locally 'residually finite' group, then by ([7], Theorem 3), G satisfies a positive law, which contradicts the assumption.

If G is not locally residually finite group, then it contains a finitely generated subgroup, which is not residually finite. We assume, that G itself is finitely generated and the intersection of all subgroups of finite index in G, denoted by N, is nontrivial. Since G/N is finitely generated, residually finite and collapsing, it must be nilpotent-by-finite by [11].

Since G does not contain free non-cyclic subsemigroups, we apply Lemma 1, then N is finitely generated. As a subgroup in the locally graded group, N must contain a proper subgroup of finite index. Then by ([6] p.196), N contains a proper characteristic subgroup $K\subsetneqq N$ of finite index in N, which is normal in G. So N/K is finite, $(G/K)/(N/K)\cong G/N$ is nilpotent-by-finite and hence G/K is finite-by-nilpotent-by-finite. Then by Lemma 2, G/K is nilpotent-by-finite and by [4], G/K is residually finite. So the intersection of all normal subgroups of finite index in G is in K. That is $N\subseteq K$, which together with $K\subsetneqq N$, gives the required contradiction. □

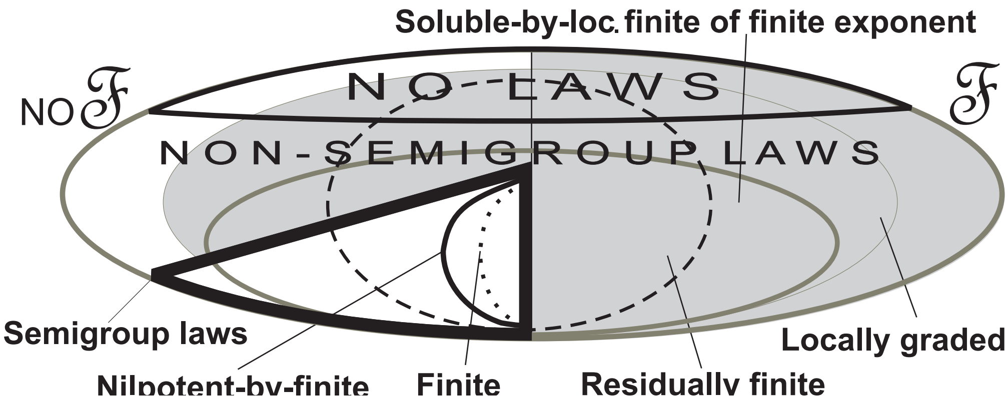
Corollary A locally graded group, which is not positively discriminating, must be nilpotent-by-locally finite of finite exponent.

Proof If a locally graded group G is not positively discriminating, then by Theorem 2, G must satisfy a positive law. By ([2] Theorem B, corrected in [3]), the locally graded group, which satisfies a positive law, must be nilpotent-by-locally finite of finite exponent. □

We show the region of known positively discriminating groups on the Groupland-map, introduced in [8]. It shows mutual relations of different properties of groups. For example, the left half of the picture contains groups without free non-cyclic subsemigroups, and the right half – groups containing free non-cyclic subsemigroups. There are three disjoint regions of groups with positive laws, non-positive laws and without laws. The locally graded groups are in the biggest inner ellipse.

By Proposition 1, the right half, and by Theorem 2 part of the left half of Groupland consist of positively discriminating groups. These regions of positively discriminating groups are marked grey.

GROUPLAND



More details on Groupland can be found via http://www.google.pl

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