What do the Engel laws and positive laws have in common

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Аннотация

Статья связана с вопросом Р. Бернса: *Что общего имеют Энгелевые и полугрупповые тождества, заставляя конечно порожденные локально ступенчатые группы содержать нильпотентную подгруппу конечного индекса?* Мы показываем, что Энгелевые и полугрупповые тождества имеют одинаковую так называемую Энгелевую конструкцию, а каждая конечно порожденная локально ступенчатая группа удовлетворяющая тождеству с такой конструкцией должна содержать нильпотентную подгруппу конечного индекса.

Abstract

The work is inspired by a question of R.Burns: *What do the Engel laws and positive laws have in common that forces finitely generated locally graded groups satisfying them to be nilpotent-by-finite?* The answer is that these laws have the same so called the Engel construction.

Introduction

Let $F=\langle x, y\rangle$ be a free group of rank 2, u be a word, and S be a subset in F.

Definition 1. *We say that a law $w \equiv 1$ has construction $u \,\widetilde{\in}\, S$ if it is equivalent to a law $u \equiv s$ for some $s \in S$.*

The laws with the same construction have similar properties. For example, the laws with construction $[x,y]\,\widetilde{\in}\,F''$ force the groups satisfying them to have perfect commutator subgroups.

We denote $x^{y^i} = y^{-i}x\,y^i$, $[x,y] = x^{-1}y^{-1}xy$, $[x,\,_iy]$ is an Engel commutator $[...[[x,y],y],...,y]$ where y is repeated i times, and $[x,\,_0y] = x$. By E_n we denote the following subgroup generated by the Engel commutators:

$$E_n = \langle\, [x,\,_iy],\ 0 \leq i \leq n \rangle.$$

We show that every binary commutator law is equivalent to a law with the following so called **Engel construction**

$$[x,y]^{k_1}\,[x,\,_2y]^{k_2}...\,[x,\,_ny]^{k_n}\ \widetilde{\in}\ E'_n.$$

Let $w \equiv 1$ be a binary law and $\mathfrak{V}$ be a variety, it defines. We prove that

- Each finitely generated group in $\mathfrak{V}$ has finitely generated commutator subgroup if and only if the law $w \equiv 1$ implies a law with the following Engel construction

$$[x,\,_ny]\,\widetilde{\in}\,E_{n-1}. \qquad (1)$$

- Positive laws and the Engel laws have the Engel construction (1). The law $x^k \equiv 1$ implies a law with the Engel construction (1).

- Each finitely generated locally graded group satisfying a law with the Engel construction (1) is nilpotent-by-finite.

The Engel construction of laws

We show that every binary commutator law is equivalent to a law $w \equiv 1$, where for some n, the word w is a product of the Engel commutators $[x,\,_iy]$, $1 \leq i \leq n$.

Theorem 1. *Every binary commutator law $w \equiv 1$ has the following Engel construction*

$$[x,y]^{k_1}\,[x,\,_2y]^{k_2}...\,[x,\,_ny]^{k_n}\ \widetilde{\in}\ E'_n, \quad k \geq 0,\ k_i \in \mathbb{Z}. \qquad (2)$$

Proof. Let $w \equiv 1$ be a commutator law. Note that F' belongs to the normal closure of x in F which is freely generated by all conjugates x^{y^i}, $i \in \mathbb{Z}$. So w is a product of some x^{y^i} with say, $-m \le i \le -m+n$. Conjugation by y^m gives us the equivalent law with $w \in \langle x^{y^i},\ 0 \le i \le n \rangle$. In this subgroup we can replace the free generators x^{y^i} by $x^{-1}x^{y^i} = [x, y^i]$, then

$$w \in \langle x^{y^i},\, 0 \le i \le n \rangle = \langle x,\ [x, y^i],\, 1 \le i \le n \rangle. \tag{3}$$

We show by induction that $\langle x,\, [x, y^i],\, 1 \le i \le n \rangle \subseteq E_n$ by proving that for $k > 0$, $[x, y^k] \in E_{k-1}[x,\, _ky]$. For $k = 1$ it is clear. Assume now that $[x, y^k] \in E_{k-1}[x,\, _ky]$. If replace $x \to [x, y]$ then

$$[[x, y], y^k] \in E_k[x,\, _{k+1}y]. \tag{4}$$

By applying the assumption and its consequence to the commutator identity

$$[x, y^{k+1}] = [x, y^k]\,[x, y]\,[[x, y], y^k], \tag{5}$$

we obtain for $k \ge 0$,

$$[x, y^{k+1}] \in E_k[x,\, _{k+1}y]. \tag{6}$$

So in view of (3),

$$w \in \langle x,\ [x, y^i],\, 1 \le i \le n \rangle \subseteq E_n.$$

Hence every commutator law is equivalent to a law $w \equiv 1$, where for some n, the word w is a product of the Engel commutators $[x,\, _iy]$, $1 \le i \le n$.
By ordering these factors *modulo* E'_n, we get the law with construction

$$[x, y]^{k_1}\,[x,\, _2y]^{k_2} \ldots\, [x,\, _ny]^{k_n} \,\widetilde{\in}\, E'_n, \quad k_i \in \mathbb{Z},\ n \ge 0.$$

□

The Milnor property and $\mathfrak{R}$-laws

To consider a special kind of laws, we recall the definition of the Milnor property of groups, the name of which was suggested by F. Point in [11].

Definition 2. *A group G satisfies the Milnor property if for all elements g, $h \in G$ the subgroup $\langle h^{-i} g\, h^i,\ i \in \mathbb{Z} \rangle$ is finitely generated.*

Note that the group $\langle h^{-i} g\, h^i,\ i \in \mathbb{Z}\rangle$ is invariant for conjugation by h, hence if it is finitely generated then it is equal to $\langle h^{-i} g\, h^i,\ i \in \mathbb{N}\rangle$.
The name of the property was motivated by result of Milnor ([8] Lemma 3) who proved that if a finitely generated group G has this property and A is an abelian normal subgroup in G so that G/A is cyclic then A is finitely generated. In 1976 Rosset noticed that the assumption that A is abelian can be dropped and proved the following results which we present in the following Lemma.

Lemma 1. *Let G be a finitely generated group satisfying the Milnor property.*
(i) Then G' is finitely generated.
(ii) If G/N is cyclic then N is finitely generated.
(iii) If G/N is polycyclic then N is finitely generated.

Proof. For (i) and (ii) see ([12] Lemmas 2,3), ([7] Lemma 3, Corollary 4). Note that in [7] the groups satisfying the Milnor property are called *restrained*. For (iii), if G/N is polycyclic then there is a finite subnormal series with cyclic factors $G = N_0 \rhd N_1 \rhd \cdots \rhd N_m = N$. Then by means of m successive applications of (ii) we conclude that N is finitely generated. $\square$

We introduce a class of laws which we call the $\mathfrak{R}$-laws (*restraining laws*) because, as we show, every group satisfying the $\mathfrak{R}$-law has the Milnor property (is *restrained*).

Definition 3. *A law is called an $\mathfrak{R}$-law if it implies a law with the following Engel construction where $k_i \in \mathbb{Z},\ n \geq 1$.*

$$[x, y]^{k_1} [x,\, _2y]^{k_2} \ldots [x,\, _{n-1}y]^{k_{n-1}} [x,\, _ny] \,\widetilde{\in}\, E'_{n-1}. \tag{7}$$

Example 1. *It is clear that an n-Engel law $[x,\, _ny] \equiv 1$ is the $\mathfrak{R}$-law.*

Example 2. *Each law of the form $x^k \equiv 1$ is the $\mathfrak{R}$-law because it implies the law $[x, y^k] \equiv 1$ which in view of (6) implies a law of the form (7).*

Theorem 2. *A law is an $\mathfrak{R}$-law if and only if every group satisfying this law has the Milnor property.*

Proof. We denote $P_k = \langle x,\ x^{y^i},\ 1 \leq i \leq k\ \rangle$ and show that $[x,\, _ky] \in P_{k-1} x^{y^k}$. For $k = 1$ it is clear. Assume that $[x,\, _ky] \in P_{k-1} x^{y^k}$, then

$$[x,\, _{k+1}y] \in (P_{k-1} x^{y^k})^{-1} (P_{k-1} x^{y^k})^y \subseteq P_k x^{y^{k+1}}.$$

It follows for $k \geq 0$ that $E_k \subseteq P_k$ which implies the equality $E_k = P_k$, because

$$E_k \subseteq P_k = \langle x,\ x^{y^i},\ 1 \leq i \leq k\ \rangle \overset{(3)}{=} \langle x,\ [x, y^i],\ 1 \leq i \leq k\ \rangle \overset{(6)}{\subseteq} E_k.$$

Hence the construction $[x,\ _n y] \,\widetilde{\in}\, E_{n-1}$ is equivalent to $x^{y^n} \,\widetilde{\in}\, P_{n-1}$, that is

$$x^{y^n} \,\widetilde{\in}\, \langle\, x,\ x^y, x^{y^2}, \ldots,\ x^{y^{n-1}} \rangle. \tag{8}$$

We use conjugation by y^{-n}, so each $\mathfrak{R}$-law has also construction

$$x \,\widetilde{\in}\, \langle\, x^{y^{-n}},\ x^{y^{-(n-1)}}, \ldots, x^{y^{-2}}, x^{y^{-1}} \rangle, \tag{9}$$

and if change $y \to y^{-1}$ we have

$$x \,\widetilde{\in}\, \langle\, x^{y},\ x^{y^2},\ \ldots, x^{y^{(n-1)}}, x^{y^n} \rangle. \tag{10}$$

Let G be a relatively free group, freely generated by $a, b, \ldots$, satisfying an $\mathfrak{R}$-law. Then (10) implies

$$a \in \langle\, a^b,\ a^{b^2},\ \ldots, a^{b^{(n-1)}}, a^{b^n} \rangle, \tag{11}$$

We conjugate (11) by b^{-1}, then

$$a^{b^{-1}} \in \langle\, a,\ a^b, , \ldots, a^{b^{(n-2)}}, a^{b^{(n-1)}} \rangle \overset{(11)}{\subseteq} \langle\, a^b,\ a^{b^2},\ \ldots, a^{b^{(n-1)}}, a^{b^n} \rangle.$$

By repeating the conjugation by b^{-1} we obtain for all $i > 0$,

$$a^{b^{-i}} \in \langle\, a^b,\ a^{b^2},\ \ldots, a^{b^{(n-1)}}, a^{b^n} \rangle. \tag{12}$$

Similarly, by (9),

$$a \in \langle\, a^{b^{-n}},\ a^{b^{-(n-1)}}, \ldots, a^{b^{-2}}, a^{b^{-1}} \rangle. \tag{13}$$

Conjugation by b gives $a^b \in \langle\, a^{b^{-n+1}},\ a^{b^{-n}}, \ldots, a^{b^{-1}}, a \rangle \overset{(13)}{\subseteq} \langle\, a^{b^{-n}},\ a^{b^{-n+1}}, \ldots, a^{b^{-1}} \rangle$. By repeating conjugation we obtain for all $i > 0$, $a^{b^i} \in \langle\, a^{b^{-n}},\ a^{b^{(-n+1)}}, \ldots,\ a^{b^{-1}} \rangle$, which, together with (12) finally gives that the subgroup

$$\langle b^{-i} a\, b^i,\ \ i \in \mathbb{Z} \rangle = \langle\, a^{b^{-n}}, a^{b^{-(n-1)}}, \ldots,\ a^{b^{-1}},\ a,\ a^b, \ldots, a^{b^{n-1}}, a^{b^n} \rangle \tag{14}$$

is finitely generated. Since for all elements g, h in any group satisfying the $\mathfrak{R}$-law, the subgroup $\langle h^{-i} g\, h^i,\ i \in \mathbb{Z} \rangle$ is an image of $\langle\, b^{-i} a\, b^i,\ i \in \mathbb{Z} \,\rangle$, we conclude that the $\mathfrak{R}$-law implies the Milnor property.

Conversely, let G be a relatively free group with free generators a, b. If the subgroup $\langle b^{-i} a\, b^{i},\quad i \in \mathbb{Z}\rangle$ is finitely generated then there exists n such that the condition (14) holds. Conjugation by b^n implies that

$$\langle b^{-i} a\, b^{i},\quad i \in \mathbb{Z}\rangle = \langle\, a, a^{b}, a^{b^2}, \ldots,\, a^{b^{2n}} \rangle = \langle b^{-i} a\, b^{i},\quad i \in \mathbb{N}\rangle. \qquad (15)$$

So we have

$$a^{b^{2n+1}} \in \langle\, a, a^{b}, a^{b^2}, \ldots,\, a^{b^{2n}} \rangle.$$

Since each relator on free generators is a law (see [9] 13.21), G satisfies a law with construction of the form (8) which defines the $\mathfrak{R}$-laws. ☐

Theorem 3. *A law is an $\mathfrak{R}$-law if and only if every finitely generated group satisfying this law has a finitely generated commutator subgroup.*

Proof. If G satisfies an $\mathfrak{R}$-law then by Theorem 2, G has the Milnor property an hence by Lemma 1 (i), G' is finitely generated.

Conversely, let G be a relatively free group defined by a law $w \equiv 1$, with free generators a, b and let G' be finitely generated. Then the normal closure of a is equal to $\langle b^{-i} a\, b^{i},\quad i \in \mathbb{Z}\rangle = \langle a\rangle\big[\langle a\rangle, \langle b\rangle\big] = \langle a\rangle G'$, hence is finitely generated. Then for some n the condition (14) holds. It follows as above, that G satisfies the Milnor property and then by Theorem 2, it satisfies an $\mathfrak{R}$-law. ☐

Positive laws are the laws of the form $u(x_l, x_2, \ldots. x_n) = v(x_l, x_2, \ldots x_n)$, where u, v are distinct words in the free group $\langle x_l, x_2 \ldots .\rangle$, written without negative powers of $x_l, x_2, \ldots x_n$.

Example 3. *Each positive law is an $\mathfrak{R}$-law.*

Proof. Each positive law implies a binary positive law if substitute $x_i \to xy^i$. It was shown by many authors (e.g. [6], [7], [11], [2] p.520) that if a group G satisfies a binary positive law then G has the Milnor property. Thus by Theorem 2, positive laws are the $\mathfrak{R}$-laws. ☐

Example 4. *For all prime p, the variety $\mathfrak{A}_p\mathfrak{A}$, where $\mathfrak{A}$ is the variety of all abelian groups, and $\mathfrak{A}_p$ – of all abelian groups of exponent p, does not satisfy an $\mathfrak{R}$-law.*

Proof. The variety $\mathfrak{A}_p\mathfrak{A}$ contains a 2-generator group $W := C_p wr C$, the wreath product of a cyclic of order p group $C_p = \langle a\rangle$ and an infinite cyclic group $C = \langle b\rangle$. The commutator subgroup W' of this group contains elements $a^{-1}a^{b^i}$ for all $i \in \mathbb{Z}$, hence W' has an infinite support and cannot be finitely generated. So by Theorem 2, $\mathfrak{A}_p\mathfrak{A}$ does not satisfy an $\mathfrak{R}$-law. ☐

A finitely generated residually finite group satisfying either an Engel law or a positive law is nilpotent-by-finite. It was proved for the Engel laws in [14] and for positive laws in [13]. By Examples 2 and 3, the Engel laws and positive laws are the $\mathfrak{R}$-laws. The following lemma extends the statement to the class of $\mathfrak{R}$-laws.

Lemma 2. *Every finitely generated residually finite group satisfying an $\mathfrak{R}$-law is nilpotent-by-finite.*

Proof. It follows from ([3] Theorem A) that if a law $w \equiv 1$ forces every *finitely generated metabelian* group satisfying this law to have a nilpotent (of class c, say) subgroup of finite index (e, say), then the same holds for every group in the class containing in particular all *residually finite* groups. Moreover, the parameters c, e depend on the law only.

So it suffices to show that every finitely generated metabelian group satisfying an $\mathfrak{R}$-law is nilpotent-by-finite. Let G be a finitely generated soluble (in particular metabelian) group satisfying an $\mathfrak{R}$-law. By Groves ([5] Theorem C), G is either nilpotent-by-finite or $var\, G$ contains a subvariety $\mathfrak{A}_p\mathfrak{A}$. Since the latter is not possible in view of Example 4, the proof is complete. □

The next property of $\mathfrak{R}$-laws concerns a finite residual R in a group G, that is the intersection of all subgroups of finite index in G.

Theorem 4. *Every finitely generated group G satisfying an $\mathfrak{R}$-law has its finite residual R finitely generated.*

Proof. By assumption the group G/R satisfies an $\mathfrak{R}$-law, hence by Theorem 2 it has the Milnor property. Then by Lemma 2, G/R is nilpotent-by-finite.

So G/R contains a nilpotent subgroup H/R of finite index. Now, since $|G : H| = |(G/R) : (H/R)| < \infty$ and G is finitely generated, both H and H/R are finitely generated. Being a finitely generated nilpotent group, H/R is polycyclic (see [9] 31.12). Since H/R also has the Milnor property, we conclude by Lemma 1 (*iii*) that R is finitely generated. □

$\mathfrak{R}$-laws and locally graded groups

The common property of the Engel laws and positive laws of being the $\mathfrak{R}$-laws is necessary and sufficient to answer why they force finitely generated locally graded groups satisfying them to be nilpotent-by-finite.

We recall that a group G is called *locally graded* if every nontrivial, finitely generated subgroup of G has a proper normal subgroup of finite index. The class of locally graded groups is closed under taking subgroups, extensions and groups which are locally-or-residually 'locally graded'. The class of locally graded groups was introduced in 1970 by S.N.Černikov [4] to avoid groups such as infinite Burnside groups or Ol'shanskii-Tarski monsters.

We can prove now the following

Theorem 5. *Every finitely generated locally graded group satisfying an $\mathfrak{R}$-law is nilpotent-by-finite.*

Proof. Let G be a finitely generated locally graded group. By Theorem 4, its finite residual R is finitely generated. Then, since G is locally graded, if $R \neq 1$, it must contain a proper subgroup (hence a proper normal subgroup) of finite index $T \subsetneq R$, say. Then by ([9], 41.43), T contains a subgroup K of finite index in R and fully invariant in R, $K \subseteq T \subsetneq R$. Thus K is normal in G. Now, since R/K is finite and G/R is nilpotent-by-finite, the isomorphism $(G/K)/(R/K) \cong G/R$ implies that G/K is finite-by-(nilpotent-by-finite). Since finite-by-nilpotent group is nilpotent-by-finite, whence G/K is nilpotent-by-finite and then residually finite. It implies $R \subseteq K$, which contradicts to $K \subseteq T \subsetneq R$. Hence $R = 1$.

So G is residually finite and by Lemma 2 is nilpotent-by-finite as required. □

Moreover, let $\mathfrak{N}_c$ denote the variety of all nilpotent groups of class $\leq c$ and $\mathfrak{B}_e$ – the variety consisting of all locally finite groups of exponent dividing e. (Note that the fact that the class $\mathfrak{B}_e$ is actually a variety, is a consequence of Zelmanov's solution of the restricted Burnside problem.) Then by result in [3] (see the proof of Lemma 2) we obtain

Corollary 1. *For every $\mathfrak{R}$-law there exist positive integers c and e depending only on the law, such that every locally graded group satisfying this law lies in the product variety $\mathfrak{N}_c\mathfrak{B}_e$*

Note Outside of the class of locally graded groups there are finitely generated groups satisfying $\mathfrak{R}$-laws (in particular, positive laws), which are not nilpotent-by-finite. For example a free Burnside group $B(r, n), r > 1$ satisfies the $\mathfrak{R}$-law $x^n \equiv 1$. If n is sufficiently large the group is infinite by results of Novikov and Adian (see [1]), hence it is not nilpotent-by-finite. Note also that a free finitely generated group satisfying the $\mathfrak{R}$-law $xy^n = y^n x$ is not nilpotent-by-finite for n sufficiently large.

Another example was given by Ol'shanskii and Storozhev in [10].

Problem Is there an $\mathfrak{R}$-law that implies neither positive nor Engel law?

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