ON FINITE NUMBER OF CONJUGACY CLASSES IN GROUPS

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Abstract. The work is inspired by an article of M. Herzog, P. Longobardi, and M. Maj, who considered groups with a finite number of infinite conjugacy classes. Their main results were obtained under assumption that the FC-center is of finite index in the group. We consider here infinite groups with a finite number of conjugacy classes of any size (FNCC-groups). Hence the FC-center in our case will be finite, but of infinite index in the group. Among results on these groups we give a criterion for a wreath product of FNCC-groups to be an FNCC-group.

1. Introduction

Many authors considered groups with some restrictions on conjugacy classes. Groups with conjugacy classes only of finite size, known as FC-groups, are well described e.g. in [3, 15, 17]. The generalization suggested in [8] releases definition of FC-groups by permitting a finite number of the infinite size conjugacy classes. In this paper we consider groups with a finite number of conjugacy classes of any size. They were called CF-groups in [11]. However, since 'CF' has many other meanings, we shall call these groups FNCC-groups.

Definition 1.1. A group is called FNCC-group if it has only Finite Number of Conjugacy Classes.

Clearly every finite group is an FNCC-group, while the infinite cyclic group is not an FNCC-group. The groups considered in [8], apart from Theorem 1.1(b), are FNCC-groups only in the case when they are finite. However, we are interested in infinite FNCC-groups.

The FNCC-groups, without special name, appear for example in [15, p. 129], [4, 5, 9], in [10, Problem 9.10] and in [14].

In 1949 the first example of an infinite FNCC-group was given in [9] by G. Higman, B.H. Neumann and H. Neumann. By means of the famous HNN-extension they proved that every torsion-free group can be embedded into a group with only two conjugacy classes. In 1952 this result was generalized by Yu. N. Gorchinskiï, who gave a construction of groups with exactly \( n \) conjugacy classes for every \( n \geq 2 \) [4, Corollary...]


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2]. However all these groups are infinitely generated, obtained as the unions of infinite sequences of $HNN$-extensions (or similar).

S.V. Ivanov proved (see [13] Thm. 41.2) that for any large enough prime $p$ there exists a 2-generator infinite group of exponent $p$ with exactly $p$ conjugacy classes, hence an $FNCC$-group.

The results of D.V. Osin from [14] imply that any countable group with only finitely many orders of elements can be embedded into a 2-generator $FNCC$-group where any two elements of the same order are conjugate. This proves existence of finitely generated infinite $FNCC$-groups with $n$ ($n \geq 2$) conjugacy classes.

The importance of finitely generated $FNCC$-groups is shown in [11, Theorem 5.2] in connection with a criterion for a group of units in a group ring to be finitely generated. In the same paper it is shown, that every $FNCC$-subgroup of units in any associative ring with polynomial identity must be finite.

We describe here some properties of $FNCC$-groups concerning their subgroups, extensions and wreath products. Some results forcing finiteness of $FNCC$-groups will also be given. In this way we give a partial answer to Question 12 posed in [11].

Our notation will be rather standard, as for example in [16]. If $G$ is a group then for elements $a, b \in G$ we denote $[a, b] = a^{-1}b^{-1}ab = a^{-1}a^{-1}ab$.

- $a^G$ denotes the conjugacy class of the element $a$ in the group $G$.
- $FC(G)$ – the FC-center of a group $G$, which consists of all elements $a \in G$ with $|a^G| < \infty$.
- $R(G)$ – the finite residual, which is the intersection of all subgroups of finite index in the group $G$.
- $C_G(R)$ – the centralizer of $R$ in the group $G$.
- A group is called anti-finite if it has no proper subgroups of finite index.
- A group is called locally graded if it has no finitely generated anti-finite subgroups.

2. Basic properties of $FNCC$-groups

Certainly the class of all $FNCC$-groups is closed under homomorphisms and finite direct products.

Lemma 2.1. Let $G$ be an $FNCC$-group with $k$ conjugacy classes and let $R \subseteq G$ be a subgroup of index $m < \infty$. Then

(i) $G$ is a union of $n$ conjugacy classes with respect to $R$, where $n \leq km$;
(ii) $R$ is an $FNCC$-group.
Proof. (1) By assumptions we have \( G = \bigcup_{i=1}^{k} g_i^G = \bigcup_{j=1}^{m} b_j R, \) \( g_i, b_j \in G, \)
Then \( G = \bigcup_{i=1}^{k} \bigcup_{j=1}^{m} g_i^{(b_j R)} = \bigcup_{i=1}^{k} \bigcup_{j=1}^{m} (g_i^{b_j})^R =: \bigcup_{i=1}^{n} a_i^R, \) where \( a_i \in G. \)

(2) If \( a_i^R \cap R \neq \emptyset, \) then \( a_i^R \subseteq R. \) This, together with (i), implies that \( R \) is an FNCC-group.
\[ \square \]

The properties considered in the following Lemma are addressed below only by their numbers.

Lemma 2.2. Let \( G \) be an FNCC-group with the finite residual \( R := R(G) \) and with the FC-center \( F := F(G). \) Then:

(i) \( R \) is of finite index in \( G; \)
(ii) \( F \) is finite;
(iii) \( R \) is an anti-finite FNCC-group;
(iv) The centralizer of \( R \) in \( G \) is equal to \( F; \)
(v) If \( G \) is infinite, then \( R \) has a simple infinite homomorphic image;
(vi) If \( G \) is torsion, then it is a group of finite exponent.

Proof. (i) Each FNCC-group has only finite number of normal subgroups, hence \( R \) is of finite index in \( G. \)

(ii) The property follows immediately from the definition of FNCC-group.

(iii) By (i) and the definition of \( R, \) it has no proper subgroup of finite index, thus it is anti-finite. Lemma 2.1(ii), gives that \( R \) is an FNCC-group.

(iv) Note first that the centralizer \( C_G(F) = \bigcap_{f \in F} C_G(f) \) is, by (ii), a subgroup of finite index in \( G. \) Then \( C_G(F) \) contains \( R \) and hence \( C_G(R) \supseteq F. \) For the converse inclusion we take \( a \in C_G(R), \) then \( C_G(a) \supseteq R \) and \( |a^G| = |G : C_G(a)| \leq |G : R|. \) Since by (i) \( |G : R| < \infty \) we get \( a \in F, \) that is \( C_G(R) \subseteq F. \) The equality follows.

(v) If \( G \) is infinite, then by (i), \( R \) is infinite and by (iii), \( R \) is an anti-finite FNCC-group. Hence \( R \) contains a maximal normal subgroup \( N, \) and \( R/N \) is infinite simple.

(vi) An FNCC-group has only finite number of orders of elements and being torsion, it has a finite exponent. \[ \square \]

By property (vi) of the above Lemma we have that every torsion FNCC-group satisfies a law of the form \( x^n \equiv 1 \) for some \( n \geq 1. \) This is an example of so called \( \mathfrak{R} \)-law. Recall that a law \( w \equiv 1 \) is an \( \mathfrak{R} \)-law if every finitely generated group \( G \) satisfying this law has \( G' \) finitely generated, (see [12, Definition 6.2]). Positive laws and Engel laws are \( \mathfrak{R} \)-laws (see [12, Corollary 6.4]).
The following Theorem generalizes Proposition 5.5 in [11] and contains Theorem 5.6 from that paper.

**Theorem 2.3.** If $G$ is an FNCC-group, then $G$ is finite in any of the following cases:

(a) $G$ is linear over a field;
(b) $G$ is locally or residually finite;
(c) $G$ is locally or residually soluble;
(d) $G$ is locally graded and either is finitely generated, or satisfies an $R$-law, or is torsion.

**Proof.** (a) This proof, based on classical arguments of Burnside, is more detailed than that in [11]. Let $G \subseteq GL(n, K)$ be an FNCC-group. The field $K$ can be assumed algebraically closed. We proceed by induction on $n$. If $n = 1$ then $G$ is abelian, hence finite.

Let $n > 1$. Assume first, that $G$ is irreducible as the group of linear transformations of $K^n$. Since $G$ is an FNCC-group, the set of traces of all elements of $G$ is finite, because conjugate matrices have the same traces. Hence by [7, Theorem 2.3.3], $G$ is finite.

Now let $G \subseteq GL(n, K)$ be reducible. After proper choice of a base in $K^n$ we can assume that there exists $m, 1 < m < n$, such that every $g \in G$ is of the form $\begin{bmatrix} \alpha(g) & x \\ 0 & \beta(g) \end{bmatrix}$, where $\alpha(g) \in GL(m, K)$ and $\beta(g) \in GL(n - m, K)$. The functions $\alpha$ and $\beta$ are homomorphisms of groups. Thus $\alpha(G)$ and $\beta(G)$ are FNCC-groups, and hence finite, by the inductive assumption. Let $N$ be the intersection of kernels of $\alpha$ and $\beta$. Then $G/N$ is finite and $N$ is an FNCC-group, by Lemma 2.1(ii), as a subgroup of finite index in $G$. On the other hand, $N$ is soluble, and even nilpotent, as a subgroup of unitriangular matrices of the form $\begin{bmatrix} 1_m & x \\ 0 & 1_{n-m} \end{bmatrix}$. Hence $N$ and so $G$ are finite, which proves (a).

In the text below, we shall assume the contrary, that $G$ is an infinite FNCC-group and show, that it leads to a contradiction.

(b) Let $G$ be locally finite. By (vi) $G$ is of finite exponent, because it is torsion. By (v), we can assume that $G$ is simple. Hence, (compare [2, Lemma 4]) $G$ does not involve all finite groups, and then by [6, Theorem 2.6], it is linear. Now by (a), $G$ is finite. A contradiction.

Since $G$ is infinite, by (i), $R(G)$ is nontrivial. Hence $G$ cannot be residually finite.

(c) Let $G$ be locally soluble. By (v), we can also assume that $G$ is an infinite simple group. However by ([16], 12.5.2) such a group is finite. A contradiction.

Let $G$ be residually soluble. Since $G$ has only finite number of normal subgroups, and each soluble quotient of $G$ is finite (which is proved just above), we conclude that $G$ is finite. A contradiction.

(d) Let $G$ be locally graded. If $G$ is finitely generated FNCC-group then its finite residual $R$ has finite index, so is finitely generated. Since
If \( G \) satisfies an \( \mathfrak{A} \)-law then by [12, Corollary 6.8] based on [1, Theorem B], \( G \) is nilpotent-by-locally finite. Hence, being an FNCC-group, \( G \) is by (b), nilpotent-by-finite. By Lemma 2.1(ii) and (c), the nilpotent normal subgroup of finite index in \( G \) is finite. Hence \( G \) is finite. A contradiction.

If \( G \) is torsion then by (vi) it has a finite exponent. Then it satisfies an \( \mathfrak{A} \)-law and, by the above case, is finite. A contradiction. \( \square \)

In connection with the above properties the following question arises.

**Question 2.4.** Does there exist an infinite and locally graded FNCC-group?

In view of the above theorem it suffices to look for an infinitely generated locally graded FNCC-group without subgroups of finite index. Moreover, such a group, if exists, could not be periodic. In the previous section we indicated the existence of infinite simple FNCC-groups. Some of them are certainly not locally graded.

As a consequence of Lemma 2.2 we obtain

**Theorem 2.5.** An infinite FNCC-group \( G \) has a normal series

\[
G \triangleright R \triangleright Z(R), \quad \text{where } R := R(G).
\]

For this series we have

- \( G/R \) is finite,
- \( R/Z(R) \) is anti-finite FNCC-group with trivial FC-center,
- \( Z(R) = C(R) \cap R = FC(G) \cap R \) is finite abelian.

**Proof.** In view of Properties (i), (iii), \( G/R \) is finite, and \( R/Z(R) \) is anti-finite FNCC-group. We show now that \( Z(R) = FC(R) \).

Let \( a \in FC(R) \), then \( [R : C_R(a)] < \infty \). However, by (iii), \( R \) is anti-finite and it follows that \( a \in Z(R) \). This means that \( FC(R) \subseteq Z(R) \). The converse inclusion is clear, so \( FC(R) = Z(R) \). Since \( R \) is FNCC-group, \( FC(R) = Z(R) \) is finite abelian. The quotient \( R/Z(R) \) is an anti-finite FNCC-group, which implies (as above) that its center and FC-center coincide. Since by (iv), \( FC(R) \) is finite, the FC-center of \( R/Z(R) \) is trivial. \( \square \)

**Question 2.6.** Does there exist an anti-finite FNCC-group \( G \) with \( Z(G) \neq 1 \)?

3. Extensions

The question whether the class of FNCC-groups is closed for extensions is still open, however in some special cases we can give a positive answer. To give an answer for finite-by-FNCC-groups, we first prove an auxiliary result.
Lemma 3.1. Let $N$ be a finite normal subgroup in a group $G$. If $G/N$ is an FNCC-group, then the finite residual $R(G)$ is of finite index in $G$, and the FC-center $FC(G)$ is finite.

Proof. Let $G/N$ be an FNCC-group. Then, by property (i), $R(G/N)$ is of finite index in $G/N$. Let $R(G/N) = H/N$, where $N \subseteq H$. Then $|G : H| < \infty$, and hence $R \subseteq H$. On the other hand, if $X < G$ and $|G : X| < \infty$, then also $|G : NX| < \infty$, and hence $H/N = R(G/N) \subseteq NX/N$. In this way we obtain that $H \subseteq NX$. Hence, $|G/X| = |G/NX| |NX/X| \leq |G/H| |N|$, which is a common bound for indices of subgroups $X$ of finite index in $G$. Now by Third Isomorphism Theorem $|G/R| < \infty$, so $R(G)$ is of finite index in $G$.

Since $G/N$ is FNCC-group, we have by (ii), that $FC(G/N)$ is finite. The assumption $|N| < \infty$ implies $N \subseteq FC(G)$. Then $FC(G/N) = FC(G/N)$ is finite and hence $FC(G)$ is finite.\Hfill \Box

Theorem 3.2. If $N$ is a finite normal subgroup in a group $G$ and $G/N$ is an FNCC-group, then $G$ is an FNCC-group.

Proof. Let $|N| = n$ and $G/N$ be an FNCC-group. By assumption $G/N$ is a sum of say $s$ conjugacy classes, then for some $a_i \in G$

$$G = a_1^G N \cup a_2^G N \cup \ldots \cup a_s^G N.$$  

By Lemma 3.1, $R(G)$ has a finite index in $G$, hence

$$G = g_1 R \cup g_2 R \cup \ldots \cup g_l R, \quad g_i \in G.$$  

By Lemma 3.1, $FC(G)$ is finite. Similarly as in the proof of (iv) we can get that $R(G)$ centralizes $FC(G)$ and hence $FC(G)$ centralizes $R(G)$. The assumption that $|N| < \infty$ implies $N \subseteq FC(G)$, thus we obtain that $N$ centralizes $R(G)$,

$$N \subseteq C_G(R).$$

We have three sets of elements:

$$\{a_i, \quad i = 1, 2, \ldots s\}, \quad \{g_j, \quad j = 1, 2, \ldots l\}, \quad N = \{x_k, \quad k = 1, 2, \ldots, n\},$$

and show that $G$ is a sum of a finite number of conjugacy classes $(a_i^g x_k)^G$. It suffices to check that each element in $G$ is in such a class. Let $b \in G$, then there is $a_i$, $g \in G$ and $x_k \in N$ such that $b = a_i^g x_k$. Moreover, there is $g_j \in G$ such that $g \in g_j R$. Then since $N \subseteq C_G(R)$,

$$b = a_i^g x_k \subseteq a_i^g R x_k \subseteq (a_i^g x_k)^R \subseteq (a_i^{g_j} x_k)^G.$$  

Hence $G$ is an FNCC-group with no more then $sln$ conjugacy classes.\Hfill \Box

Theorem 3.3. Let $R$ be a normal subgroup of finite index in a group $G$.

If $R$ is an FNCC-group and every inner automorphism of $G$ restricted to $R$ is an inner automorphism of $R$, then $G$ is an FNCC-group.
Proof. Let $\varphi : G \to Aut(R)$ be a homomorphism given by
$$\varphi(g) : r \to r^g \quad \text{for} \quad g \in G \quad \text{and} \quad r \in R.$$ From the assumption on automorphisms we have that $\varphi(G) = \varphi(R)$ is an FNCC-group, as a homomorphic image of $R$. The kernel of $\varphi$ is equal to $C = C_G(R)$. The subgroup $C \cap R$ is finite, because $R$ is an FNCC-group. The group $C/(C \cap R) \simeq CR/R \subseteq G/R$ is finite, by assumption on $R$. Thus $C$ is finite. Now, $G/C \simeq \varphi(G)$ is an FNCC-group and hence, by Theorem 3.2, $G$ is an FNCC-group. □

The following question is natural is this place

**Question 3.4.** Let $R$ be a normal subgroup of finite index in a group $G$. Assume that $R$ is an FNCC-group and every inner automorphism of $G$ restricted to $R$ preserves conjugacy classes in $R$. Is $G$ an FNCC-group?

Now we show that to speak of a finite extension of an FNCC group, it suffices to consider only finite cyclic extensions.

**Lemma 3.5.** Let $G$ be a group and let $G = \bigcup_{j=1}^{m} G_j$, where $G_j \subseteq G$ for $j = 1, \ldots, m$. If the subgroups $G_j$ are FNCC-groups then $G$ is an FNCC-group.

**Proof.** Let each $G_j$ be an FNCC-group. By assumption there are elements $a_{j1}, a_{j2}, \ldots, a_{jn_j} \in G_j$ such that
$$G_j = \bigcup_{i=1}^{n_j} (a_{ji})^{G_j}, \text{ which implies } G = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} (a_{ji})^{G_j},$$ and hence $G$ is an FNCC-group. □

Let $R \subseteq G$ be a normal subgroup of finite index. Then there are elements $a_1, \ldots, a_n \in G$ such that
$$G = \bigcup_{i=1}^{n} \langle R, a_i \rangle = \bigcup_{i=1}^{n} G_i, \text{ where } G_i = \langle R, a_i \rangle. \quad (1)$$

Now for $G_i$ we have a number $l_i$ and a normal series
$$R = H_{i0} \leq \ldots \leq H_{il_i} = G_i \quad \text{such that all factor groups } H_{ij}/H_{ij-1} \text{ are cyclic of prime orders.} \quad (2)$$

Under notation from (1) and (2) in Lemma 3.5 we have

**Corollary 3.6.** Let $R \subseteq G$ be a normal subgroup of finite index. If $R$ is an FNCC-group then $G$ is an FNCC-group if and only if groups $\langle R, a_i \rangle$ are FNCC-groups, or equivalently, groups $H_{ij}$ are FNCC-groups.
Remark. If one is interested only in semidirect product of the type $G = R \rtimes F$, where $F$ is a finite group then, by (1), it suffices to consider only the case when $F$ is cyclic, hence it is a direct product of cyclic $p$-groups. Thus one can restrict to the case when $F$ is cyclic of prime power order, because for $F = F_1 \times F_2$ we have in a natural way the following formula:

\[ R \rtimes (F_1 \times F_2) \simeq (R \rtimes F_1) \rtimes F_2. \]

Now we concentrate on a special type of semidirect products.

Lemma 3.7. Let $P$ be an FNCC-group and $R = \prod P_i$ - the direct product of $n$ copies of $P$. Let $(b)_n$ be a cyclic group of order $n$, and $G = R \rtimes (b)_n$ - the semidirect product, where $b$ acts on $R$ as a cyclic permutation of factors. Then the coset $bR$ is contained in a union of finite number of conjugacy classes in $G$.

Proof. The set of representatives of conjugacy classes in $P$ we denote by $\mathcal{P} := \{p_1, p_2, \ldots, p_m\}$. Then the elements in $R$ are of the form

\[ (p_{a_1}^{p_1}, p_{a_2}^{p_2}, \ldots, p_{a_n}^{p_n}) = (p_{j_1}, p_{j_2}, \ldots, p_{j_n})^{(a_1, a_2, \ldots, a_n)}, \quad p_j \in \mathcal{P}, \quad a_j \in R. \]

Hence $R$ has $m^n$ conjugacy classes with the representatives

\[ \rho = (p_{j_1}, p_{j_2}, \ldots, p_{j_n}), \quad p_j \in \mathcal{P}. \]

To prove the Lemma we show that each element $br \in bR$, is in some of $m^n$ conjugacy classes of the form $(bp_z)^G$. It suffices to find for each $r \in R$ such $x \in R$ and $\rho_z$ of the form (5), that the following equality holds $br = (bp_z)^x$. In view of the identity

\[ (bp)^x = x^{-1}(bp)x = b b^{-1} x^{-1}(bp)x = b(x^{-1} \rho x, \rho \rho z) \]

the equality $br = (bp_z)^x$ can be written as

\[ r = (x^{-1} \rho z, \rho z) \]

where $r$ is any given element of the form (4), with the unknown elements $x = (x_1, x_2, \ldots, x_n) \in R$, and $\rho_z$ of the form (5). We shall find a solution where $\rho_z$ is of the form

\[ \rho_z = (p_z, e, e, \ldots, e). \]

Since $(x^{-1})^{-1} = (x_n^{-1}, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$, the equation (6) is now:

\[ (p_{a_1}^{p_1}, p_{a_2}^{p_2}, \ldots, p_{a_n}^{p_n}) = (x_n^{-1}, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})(p_z, e, e, \ldots, e)(x_1, x_2, \ldots, x_n). \]

It implies $n$ equations on elements of $P$ with unknowns $p_z, x_1, x_2, \ldots, x_n$.

\[ p_{a_1}^{p_1} = x_n^{-1} p_z x_1, \quad p_{a_2}^{p_2} = x_1^{-1} x_2, \quad p_{a_3}^{p_3} = x_2^{-1} x_3, \ldots, p_{a_n}^{p_n} = x_{n-1}^{-1} x_n. \]

If multiply these equations by sides, we get

\[ p_{a_1}^{p_1} p_{a_2}^{p_2} p_{a_3}^{p_3} \cdots p_{a_n}^{p_n} = x_n^{-1} p_z x_n = p_z x_n. \]
Then, since $p_{j_1}^{a_1}p_{j_2}^{a_2}p_{j_3}^{a_3}\ldots p_{j_n}^{a_n}$ is an element in $P$, it is equal to some $p_j^u$, where $u \in R$. So we get the solution: $p_z = p_j$ and $x_n = u$. Then $\rho_z = (p_j, e, e, \ldots, e)$, and by (8),
\[x_1 = p_j^{-1}u p_{j_1}^{a_1}, \quad x_2 = x_1p_{j_2}^{a_2} = p_j^{-1}u p_{j_1}^{a_1}p_{j_2}^{a_2}, \ldots,
\[x_i = x_{i-1}p_{j_i}^{a_i} = p_j^{-1}u p_{j_1}^{a_1}p_{j_2}^{a_2}\ldots p_{j_{i-1}}^{a_{i-1}}p_{j_i}^{a_i}, \quad i < n.
\]
which finishes the proof. \qed

**Theorem 3.8.** Let $P$ be an FNCC-group, $n \geq 1$, and $R = \prod_i^n P_i$, be the direct product of $n$ copies of $P$. Let $B \subseteq S_n$ be a group permuting factors in $R$. Then the semidirect product $G = R \rtimes B$ is an FNCC-group.

**Proof.** Lemma 2.1(ii) applied to $G \subseteq R \rtimes S_n$ allows us to prove the result only for $B = S_n$. We are going to proceed by induction on $n$. For $n = 1$ we have $G \simeq P$ and the result is trivial.

Now let $n > 1$. If $\sigma \in S_n$ is a cycle of length $n$ then, by Lemma 3.7, the coset $\sigma R$ is contained in a union of finite number of conjugacy classes. If $\sigma$ is not of such type then we can write $\sigma = \gamma \delta$, where $\gamma$ permutes cyclically $m < n$ factors of $R$ and is fixed on the others, while $\delta$ permutes at most $n - m$ factors of $R$, fixed by $\gamma$. Then $\gamma \subseteq S_m$ permutes factors of $P^m$ and $\delta \subseteq S_{n-m}$ permutes factors of $P^{n-m}$ in a natural way. Let $G_1$ be the semidirect product of $P^m$ with $\langle \gamma \rangle$ and $G_2$ the semidirect product of $P^{n-m}$ with $\langle \delta \rangle$ under these actions. Then, by Lemma 2.1(ii) applied to extensions $\langle P^m, \gamma \rangle \subseteq G_1$, $\langle P^{n-m}, \delta \rangle \subseteq G_2$, and by the inductive assumption, $G_1$ and $G_2$ are FNCC-groups. Thus $G_1 \times G_2$ is also an FNCC-group, contained in $G$ in a natural way. Moreover, $\sigma R \subseteq G_1 \times G_2$. This means, that $\sigma R$ is contained in a union of a finite number of conjugacy classes in $G$. Now the result follows, because $R$ is of finite index in $G$. \qed

For further text we recall that the restricted wreath product $A \wr B$ of groups $A$ and $B$ is a semidirect product $G = R \rtimes B$ where $R = \prod_{b \in B} A^b$ is the direct product of copies of $A$, numbered by elements of $B$ and $B$ acts on $R$ by shifting indices. Instead of $A^e$ we write $A$, and $A^b = b^{-1}A b$. Every element $g \in G$ can be uniquely written as $g = bw$ where $b \in B$ and $w$ is a product of commuting factors $a^b$, where $b \in B$, $a^b = b^{-1}a b$, $ba \cdot b_1a_1 = bb_1a^b_1a_1$.

Now we give a criterion for a restricted wreath product of groups to be an FNCC-group.

**Theorem 3.9.** A restricted wreath product $A \wr B$ is an FNCC-group if and only if $A$ is an FNCC-group and $B$ is finite.
Proof. Let $G = A \wr B = (\prod_{b \in B} A^b) \rtimes B = R \rtimes B$ be an $FNCC$-group. In the restricted wreath product each element $r \in R$ has a finite support of the length $s(r)$, say. Moreover, the conjugate elements have supports of the same length. If $G$ is an $FNCC$-group, then the lengths of possible supports have only finite number of values, which is possible only if the group $B$ is finite. Then $R$ is a subgroup of finite index in $G$ and by Lemma 2.1(ii), $R$ is an $FNCC$-group. Then $A$ is an $FNCC$ group as an image of $R$.

The converse implication follows from Theorem 3.8, because the group $B$ acts by permutations on the subscripts of direct factors in $R$. □

REFERENCES

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