ON FINITE NUMBER OF CONJUGACY CLASSES IN GROUPS

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Abstract. The work is inspired by an article of M. Herzog, P. Longobardi, and M. Maj, who considered groups with a finite number of infinite conjugacy classes. Their main results were obtained under assumption that the FC-center is of finite index in the group. We consider here infinite groups with a finite number of conjugacy classes of any size ($FNCC$-groups). Hence the FC-center in our case will be finite, but of infinite index in the group. Among results on these groups we give a criterion for a wreath product of $FNCC$-groups to be an $FNCC$-group.

1. Introduction

Many authors considered groups with some restrictions on conjugacy classes. Groups with conjugacy classes only of finite size, known as FC-groups, are well described e.g. in [3, 15, 17]. The generalization suggested in [8] releases definition of FC-groups by permitting a finite number of the infinite size conjugacy classes. In this paper we consider groups with a finite number of conjugacy classes of any size. They were called CF-groups in [11]. However, since 'CF' has many other meanings, we shall call these groups $FNCC$-groups.

Definition 1.1. *A group is called $FNCC$-group if it has only Finite Number of Conjugacy Classes.*

Clearly every finite group is an $FNCC$-group, while the infinite cyclic group is not an $FNCC$-group. The groups considered in [8], apart from Theorem 1.1(b), are $FNCC$-groups only in the case when they are finite. However, we are interested in infinite $FNCC$-groups.

The $FNCC$-groups, without special name, appear for example in [15, p. 129], [4, 5, 9], in [10, Problem 9.10] and in [14].

In 1949 the first example of an infinite $FNCC$-group was given in [9] by G. Higman, B. H. Neumann and H. Neumann. By means of the famous HNN-extension they proved that every torsion-free group can be embedded into a group with only two conjugacy classes. In 1952 this result was generalized by Yu. N. Gorchinskii, who gave a construction of groups with exactly n conjugacy classes for every $n \geq 2$ [4, Corollary

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2]. However all these groups are infinitely generated, obtained as the unions of infinite sequences of HNN-extensions (or similar).

S. V. Ivanov proved (see [13] Thm. 41.2) that for any large enough prime p there exists a 2-generator infinite group of exponent p with exactly p conjugacy classes, hence an $FNCC$-group.

The results of D. V. Osin from [14] imply that any countable group with only finitely many orders of elements can be embedded into a 2-generator $FNCC$-group where any two elements of the same order are conjugate. This proves existence of finitely generated infinite $FNCC$-groups with n $(n \geq 2)$ conjugacy classes.

The importance of finitely generated $FNCC$-groups is shown in [11, Theorem 5.2] in connection with a criterion for a group of units in a group ring to be finitely generated. In the same paper it is shown, that every $FNCC$-subgroup of units in any associative ring with polynomial identity must be finite.

We describe here some properties of $FNCC$-groups concerning their subgroups, extensions and wreath products. Some results forcing finiteness of $FNCC$-groups will also be given. In this way we give a partial answer to Question 12 posed in [11].
Our notation will be rather standard, as for example in [16]. If G is a group then for elements $a, b \in G$ we denote $[a, b] = a^{-1}b^{-1}a\,b = a^{-1}a^b$.

- a^G denotes the conjugacy class of the element a in the group G.
- $FC(G)$ – the FC-center of a group G, which consists of all elements $a \in G$ with $|a^G| < \infty$.
- $R(G)$ – the finite residual, which is the intersection of all subgroups of finite index in the group G.
- $C_G(R)$ – the centralizer of R in the group G.
- A group is called *anti-finite* if it has no proper subgroups of finite index.
- A group is called *locally graded* if it has no finitely generated anti-finite subgroups.

2. Basic properties of $FNCC$-groups

Certainly the class of all $FNCC$-groups is closed under homomorphisms and finite direct products.

Lemma 2.1. *Let G be an $FNCC$-group with k conjugacy classes and let $R \subseteq G$ be a subgroup of index $m < \infty$. Then*

(i) G is a union of n conjugacy classes with respect to R, where $n \leq km$;
(ii) R is an $FNCC$-group.

Proof. (1) By assumptions we have $G = \bigcup_{i=1}^{k} g_i^G = \bigcup_{j=1}^{m} b_j R,\ g_i, b_j \in G$,

Then $$G = \bigcup_{i=1}^{k}\bigcup_{j=1}^{m} g_i^{(b_j R)} = \bigcup_{i=1}^{k}\bigcup_{j=1}^{m} (g_i^{b_j})^R =: \bigcup_{i=1}^{n} a_i^R, \quad \text{where} \quad a_i \in G.$$

(2) If $a_i^R \cap R \neq \emptyset$, then $a_i^R \subseteq R$. This, together with (i), implies that R is an $FNCC$-group.

□

The properties considered in the following Lemma are addressed below only by their numbers.

Lemma 2.2. *Let G be an $FNCC$-group with the finite residual $R := R(G)$ and with the FC-center $F := F(G)$. Then:*

(i) *R is of finite index in G;*
(ii) *F is finite;*
(iii) *R is an anti-finite $FNCC$-group;*
(iv) *The centralizer of R in G is equal to F;*
(v) *If G is infinite, then R has a simple infinite homomorphic image;*
(vi) *If G is torsion, then it is a group of finite exponent.*

Proof. (i) Each $FNCC$-group has only finite number of normal subgroups, hence R is of finite index in G.

(ii) The property follows immediately from the definition of $FNCC$-group.

(iii) By (i) and the definition of R, it has no proper subgroup of finite index, thus it is anti-finite. Lemma 2.1(ii), gives that R is an $FNCC$-group.

(iv) Note first that the centralizer $C_G(F) = \cap_{f\in F} C_G(f)$ is, by (ii), a subgroup of finite index in G. Then $C_G(F)$ contains R and hence $C_G(R) \supseteq F$. For the converse inclusion we take $a \in C_G(R)$, then $C_G(a) \supseteq R$ and $|a^G| = |G : C_G(a)| \leq |G : R|$. Since by (i) $|G : R| < \infty$ we get $a \in F$, that is $C_G(R) \subseteq F$. The equality follows.

(v) If G is infinite, then by (i), R is infinite and by (iii), R is an anti-finite $FNCC$-group. Hence R contains a maximal normal subgroup N, and R/N is infinite simple.

(vi) An $FNCC$-group has only finite number of orders of elements and being torsion, it has a finite exponent. □

By property (vi) of the above Lemma we have that every torsion $FNCC$-group satisfies a law of the form $x^n \equiv 1$ for some $n \geq 1$. This is an example of so called $\mathfrak{R}$-law. Recall that a law $w \equiv 1$ is an $\mathfrak{R}$-law if every finitely generated group G satisfying this law has G' finitely generated, (see [12, Definition 6.2]). Positive laws and Engel laws are $\mathfrak{R}$-laws (see [12, Corollary 6.4]).

The following Theorem generalizes Proposition 5.5 in [11] and contains Theorem 5.6 from that paper.

Theorem 2.3. *If G is an $FNCC$-group, then G is finite in any of the following cases:*

(a) *G is linear over a field;*
(b) *G is locally or residually finite;*
(c) *G is locally or residually soluble;*
(d) *G is locally graded and either is finitely generated, or satisfies an $\mathfrak{R}$-law, or is torsion.*

Proof. (a) This proof, based on classical arguments of Burnside, is more detailed than that in [11]. Let $G \subseteq GL(n, K)$ be an $FNCC$-group. The field K can be assumed algebraically closed. We proceed by induction on n. If $n = 1$ then G is abelian, hence finite.

Let $n > 1$. Assume first, that G is irreducible as the group of linear transformations of K^n. Since G is an $FNCC$-group, the set of traces of all elements of G is finite, because conjugate matrices have the same traces. Hence by [7, Theorem 2.3.3], G is finite.

Now let $G \subseteq GL(n, K)$ be reducible. After proper choice of a base in K^n we can assume that there exists m, $1 < m < n$, such that every $g \in G$ is of the form $\left[\begin{smallmatrix} \alpha(g) & x \\ 0 & \beta(g) \end{smallmatrix}\right]$, where $\alpha(g) \in GL(m, K)$ and $\beta(g) \in GL(n-m, K)$. The functions α and β are homomorphisms of groups. Thus $\alpha(G)$ and $\beta(G)$ are $FNCC$-groups, and hence finite, by the inductive assumption. Let N be the intersection of kernels of α and β. Then G/N is finite and N is an $FNCC$-group, by Lemma 2.1(ii), as a subgroup of finite index in G. On the other hand, N is soluble, and even nilpotent, as a subgroup of unitriangular matrices of the form $\left[\begin{smallmatrix} 1_m & x \\ 0 & 1_{n-m} \end{smallmatrix}\right]$. Hence N and so G are finite, which proves (a).

In the text below, we shall assume the contrary, that G is an infinite $FNCC$-group and show, that it leads to a contradiction.

(b) Let G be locally finite. By (vi) G is of finite exponent, because it is torsion. By (v), we can assume that G is simple. Hence, (compare [2, Lemma 4]) G does not involve all finite groups, and then by [6, Theorem 2.6], it is linear. Now by (a), G is finite. A contradiction.

Since G is infinite, by (i), $R(G)$ is nontrivial. Hence G cannot be residually finite.

(c) Let G be locally soluble. By (v), we can also assume that G is an infinite simple group. However by ([16], 12.5.2) such a group is finite. A contradiction.

Let G be residually soluble. Since G has only finite number of normal subgroups, and each soluble quotient of G is finite (which is proved just above), we conclude that G is finite. A contradiction.

(d) Let G be locally graded. If G is finitely generated $FNCC$-group then its finite residual R has finite index, so is finitely generated. Since

G is locally graded, R has a subgroup of finite index, which is impossible, because by (iii) R is anti-finite.

If G satisfies an $\mathfrak{R}$-law then by [12, Corollary 6.8] based on [1, Theorem B], G is nilpotent-by-locally finite. Hence, being an $FNCC$-group, G is by (b), nilpotent-by-finite. By Lemma 2.1(ii) and (c), the nilpotent normal subgroup of finite index in G is finite. Hence G is finite. A contradiction.

If G is torsion then by (vi) it has a finite exponent. Then it satisfies an $\mathfrak{R}$-law and, by the above case, is finite. A contradiction. □

In connection with the above properties the following question arises.

Question 2.4. *Does there exist an infinite and locally graded $FNCC$-group?*

In view of the above theorem it suffices to look for an infinitely generated locally graded $FNCC$-group without subgroups of finite index. Moreover, such a group, if exists, could not be periodic. In the previous section we indicated the existence of infinite simple $FNCC$-groups. Some of them are certainly not locally graded.

As a consequence of Lemma 2.2 we obtain

Theorem 2.5. *An infinite $FNCC$-group G has a normal series*

$$G \rhd R \rhd Z(R), \quad \text{where} \quad R := R(G).$$

For this series we have

- *G/R is finite,*
- *$R/Z(R)$ is anti-finite $FNCC$-group with trivial FC-center,*
- *$Z(R) = C(R) \cap R = FC(G) \cap R$ is finite abelian.*

Proof. In view of Properties (i), (iii), G/R is finite, and $R/Z(R)$ is anti-finite $FNCC$-group. We show now that $Z(R) = FC(R)$.

Let $a \in FC(R)$, then $[R : C_R(a)] < \infty$. However, by (iii), R is anti-finite and it follows that $a \in Z(R)$. This means that $FC(R) \subseteq Z(R)$. The converse inclusion is clear, so $FC(R) = Z(R)$. Since R is $FNCC$-group, $FC(R) = Z(R)$ is finite abelian. The quotient $R/Z(R)$ is an anti-finite $FNCC$-group, which implies (as above) that its center and FC-center coincide. Since by (iv), $FC(R)$ is finite, the FC-center of $R/Z(R)$ is trivial. □

Question 2.6. *Does there exist an anti-finite $FNCC$-group G with $Z(G) \neq 1$?*

3. Extensions

The question whether the class of $FNCC$-groups is closed for extensions is still open, however in some special cases we can give a positive answer. To give an answer for finite-by-$FNCC$-groups, we first prove an auxiliary result

Lemma 3.1. *Let N be a finite normal subgroup in a group G. If G/N is an $FNCC$-group, then the finite residual $R(G)$ is of finite index in G, and the FC-center $FC(G)$ is finite.*

Proof. Let G/N be an $FNCC$-group. Then, by property (i), $R(G/N)$ is of finite index in G/N. Let $R(G/N) = H/N$, where $N \subseteq H$. Then $|G : H| < \infty$, and hence $R \subseteq H$. On the other hand, if $X \lhd G$ and $|G : X| < \infty$, then also $|G : NX| < \infty$, and hence $H/N = R(G/N) \subseteq NX/N$. In this way we obtain that $H \subseteq NX$. Hence, $|G/X| = |G/NX||NX/X| \leq |G/H|\cdot|N|$, which is a common bound for indices of subgroups X of finite index in G. Now by Third Isomorphism Theorem $|G/R| < \infty$, so $R(G)$ is of finite index in G.

Since G/N is $FNCC$-group, we have by (ii), that $FC(G/N)$ is finite. The assumption $|N| < \infty$ implies $N \subseteq FC(G)$. Then $FC(G)/N = FC(G/N)$ is finite and hence $FC(G)$ is finite. □

Theorem 3.2. *If N is a finite normal subgroup in a group G and G/N is an $FNCC$-group, then G is an $FNCC$-group.*

Proof. Let $|N| = n$ and G/N be an $FNCC$-group. By assumption G/N is a sum of say s conjugacy classes, then for some $a_i \in G$

$$G = a_1^G N \cup a_2^G N \cup, ... \cup a_s^G N.$$

By Lemma 3.1, $R(G)$ has a finite index l, say, in G, hence

$$G = g_1 R \cup g_2 R \cup, ... g_l R, \quad g_i \in G.$$

By Lemma 3.1, $FC(G)$ is finite. Similarly as in the proof of (iv) we can get that $R(G)$ centralizes $FC(G)$ and hence $FC(G)$ centralizes $R(G)$. The assumption that $|N| < \infty$ implies $N \subseteq FC(G)$, thus we obtain that N centralizes $R(G)$,

$$N \subseteq C_G(R).$$

We have three sets of elements:

$$\{a_i,\ i = 1, 2, ...s\}, \quad \{g_j,\ j = 1, 2, ...l\}, \quad N = \{x_k,\ k = 1, 2, ..., n, \},$$

and show that G is a sum of a finite number of conjugacy classes $(a_i^{g_j} x_k)^G$. It suffices to check that each element in G is in such a class. Let $b \in G$, then there is a_i, $g \in G$ and $x_k \in N$ such that $b = a_i^g x_k$. Moreover, there is $g_j \in G$ such that $g \in g_j R$. Then since $N \subseteq C_G(R)$,

$$b = a_i^g x_k \in a_i^{g_j R} x_k \subseteq (a_i^{g_j} x_k)^R \subseteq (a_i^{g_j} x_k)^G.$$

Hence G is an $FNCC$-group with no more then sln conjugacy classes. □

Theorem 3.3. *Let R be a normal subgroup of finite index in a group G. If R is an $FNCC$-group and every inner automorphism of G restricted to R is an inner automorphism of R, then G is an $FNCC$-group.*

Proof. Let $\varphi : G \to Aut(R)$ be a homomorphism given by

$$\varphi(g) : r \to r^g \quad \text{for} \quad g \in G \quad \text{and} \quad r \in R.$$

From the assumption on automorphisms we have that $\varphi(G) = \varphi(R)$ is an $FNCC$-group, as a homomorphic image of R. The kernel of φ is equal to $C = C_G(R)$. The subgroup $C \cap R$ is finite, because R is an $FNCC$-group. The group $C/(C \cap R) \simeq CR/R \subseteq G/R$ is finite, by assumption on R. Thus C is finite. Now, $G/C \simeq \varphi(G)$ is an $FNCC$-group and hence, by Theorem 3.2, G is an $FNCC$-group. □

The following question is natural is this place

Question 3.4. *Let R be a normal subgroup of finite index in a group G. Assume that R is an $FNCC$-group and every inner automorphism of G restricted to R preserves conjugacy classes in R. Is G an $FNCC$-group?*

Now we show that to speak of a finite extension of an $FNCC$ group, it suffices to consider only finite cyclic extensions.

Lemma 3.5. *Let G be a group and let $G = \bigcup_{j=1}^{m} G_j$, where $G_j \subseteq G$ for $j = 1, \ldots, m$. If the subgroups G_j are $FNCC$-groups then G is an $FNCC$-group.*

Proof. Let each G_j be an $FNCC$-group. By assumption there are elements $a_{j1}, a_{j2}, \ldots, a_{jn_j} \in G_j$ such that

$$G_j = \bigcup_{i=1}^{n_j} (a_{ji})^{G_j}, \quad \text{which implies} \quad G = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n_j} (a_{ji})^{G},$$

and hence G is an $FNCC$-group. □

Let $R \subseteq G$ be a normal subgroup of finite index. Then there are elements $a_1, \ldots, a_n \in G$ such that

$$(1) \qquad G = \bigcup_{i=1}^{n} \langle R, a_i \rangle = \bigcup_{i=1}^{n} G_i, \quad \text{where} \quad G_i = \langle R, a_i \rangle.$$

Now for G_i we have a number l_i and a normal series

$$(2) \qquad R = H_{i0} \leq \ldots \leq H_{i\,l_i} = G_i$$

such that all factor groups $H_{ij}/H_{i(j-1)}$ are cyclic of prime orders.

Under notation from (1) and (2) in Lemma 3.5 we have

Corollary 3.6. *Let $R \subseteq G$ be a normal subgroup of finite index. If R is an $FNCC$-group then G is an $FNCC$-group if and only if groups $\langle R, a_i \rangle$ are $FNCC$-groups, or equivalently, groups H_{ij} are $FNCC$-groups.*

Remark. If one is interested only in semidirect product of the type $G = R \rtimes F$, where F is a finite group then, by (1), it suffices to consider only the case when F is cyclic, hence it is a direct product of cyclic p-groups. Thus one can restrict to the case when F is cyclic of prime power order, because for $F = F_1 \times F_2$ we have in a natural way the following formula:

$$R \rtimes (F_1 \times F_2) \simeq (R \rtimes F_1) \rtimes F_2. \tag{3}$$

Now we concentrate on a special type of semidirect products.

Lemma 3.7. *Let P be an $FNCC$-group and $R = \prod_i^{n\,\times} P_i$, – the direct product of n copies of P. Let $\langle b\rangle_n$ be a cyclic group of order n, and $G = R \rtimes \langle b\rangle_n$ – the semidirect product, where b acts on R as a cyclic permutation of factors. Then the coset bR is contained in a union of finite number of conjugacy classes in G.*

Proof. The set of representatives of conjugacy classes in P we denote by $\mathcal{P} := \{p_1, p_2, \dots, p_m\}$. Then the elements in R are of the form

$$(p_{j_1}^{a_1}, p_{j_2}^{a_2}, ..., p_{j_n}^{a_n}) = \big(p_{j_1}, p_{j_2}, ..., p_{j_n}\big)^{(a_1, a_2, ..., a_n)}, \quad p_j \in \mathcal{P}, \quad a_j \in R. \tag{4}$$

Hence R has m^n conjugacy classes with the representatives

$$\rho = \big(p_{j_1}, p_{j_2}, \dots, p_{j_n}\big), \quad p_j \in \mathcal{P}. \tag{5}$$

To prove the Lemma we show that each element $br \in bR$, is in some of m^n conjugacy classes of the form $(b\rho_z)^G$. It suffices to find for each $r \in R$ such $x \in R$ and ρ_z of the form (5), that the following equality holds $br = (b\rho_z)^x$. In view of the identity

$$(b\rho)^x = x^{-1}(b\rho)x = b\,b^{-1}x^{-1}(b\rho)\,x = b\,(x^b)^{-1}\rho\,x$$

the equality $br = (b\rho_z)^x$ can be written as

$$r = (x^b)^{-1}\rho_z\,x, \tag{6}$$

where r is any given element of the form (4), with the unknown elements $x = (x_1, x_2, \dots, x_n) \in R$, and ρ_z of the form (5). We shall find a solution where ρ_z is of the form

$$\rho_z = (p_z, e, e, , \dots, e). \tag{7}$$

Since $(x^b)^{-1} = (x_n^{-1}, x_1^{-1}, x_2^{-1}, ..., x_{n-1}^{-1})$, the equation (6) is now:

$$(p_{j_1}^{a_1}, p_{j_2}^{a_2}, ..., p_{j_n}^{a_n}) = (x_n^{-1}, x_1^{-1}, x_2^{-1}, ..., x_{n-1}^{-1})(p_z, e, e, ..., e)(x_1, x_2, ..., x_n).$$

It implies n equations on elements of P with unknowns $p_z, x_1, x_2, \dots, x_n$.

$$p_{j_1}^{a_1} = x_n^{-1}p_z\,x_1, \quad p_{j_2}^{a_2} = x_1^{-1}x_2, \quad p_{j_3}^{a_3} = x_2^{-1}x_3, \ \dots\ p_{j_i}^{a_i} = x_{n-1}^{-1}x_n. \tag{8}$$

If multiply these equations by sides, we get

$$p_{j_1}^{a_1}p_{j_2}^{a_2}p_{j_3}^{a_3}\cdots\ p_{j_n}^{a_n} = x_n^{-1}p_z\,x_n = p_z^{x_n}.$$

Then, since $p_{j_1}^{a_1}p_{j_2}^{a_2}p_{j_3}^{a_3}\cdots\ p_{j_n}^{a_n}$ is an element in P, it is equal to some p_j^u, where $u \in R$. So we get the solution: $p_z = p_j$ and $x_n = u$. Then $\rho_z = (p_j, e, e, \dots, e)$, and by (8),

$$x_1 = p_j^{-1}u\,p_{j_1}^{a_1},\quad x_2 = x_1 p_{j_2}^{a_2} = p_j^{-1}u\,p_{j_1}^{a_1}p_{j_2}^{a_2}, \dots,$$

$$x_i = x_{i-1}p_{j_i}^{a_i} = p_j^{-1}u\,p_{j_1}^{a_1}p_{j_2}^{a_2}\dots p_{j_{i-1}}^{a_{i-1}}\,p_{j_i}^{a_i},\ i < n.$$

which finishes the proof. □

Theorem 3.8. *Let P be an $FNCC$-group, $n \geq 1$, and $R = \prod_i^{n\,\times} P_i$, be the direct product of n copies of P. Let $B \subseteq S_n$ be a group permuting factors in R. Then the semidirect product $G = R \rtimes B$ is an $FNCC$-group.*

Proof. Lemma 2.1(ii) applied to $G \subseteq R \rtimes S_n$ allows us to prove the result only for $B = S_n$. We are going to proceed by induction on n. For $n = 1$ we have $G \simeq P$ and the result is trivial.

Now let $n > 1$. If $\sigma \in S_n$ is a cycle of length n then, by Lemma 3.7, the coset σR is contained in a union of finite number of conjugacy classes. If σ is not of such type then we can write $\sigma = \gamma\delta$, where γ permutes cyclically $m < n$ factors of R and is fixed on the others, while δ permutes at most $n-m$ factors of R, fixed by γ. Then $\gamma \subseteq S_m$ permutes factors of P^m and $\delta \subseteq S_{n-m}$ permutes factors of P^{n-m} in a natural way. Let G_1 be the semidirect product of P^m with $\langle\gamma\rangle$ and G_2 the semidirect product of P^{n-m} with $\langle\delta\rangle$ under these actions. Then, by Lemma 2.1(ii) applied to extensions $\langle P^m, \gamma\rangle \subseteq G_1$, $\langle P^{n-m}, \delta\rangle \subseteq G_2$, and by the inductive assumption, G_1 and G_2 are $FNCC$-groups. Thus $G_1 \times G_2$ is also an $FNCC$-group, contained in G in a natural way. Moreover, $\sigma R \subseteq G_1 \times G_2$. This means, that σR is contained in a union of a finite number of conjugacy classes in G. Now the result follows, because R is of finite index in G. □

For further text we recall that the restricted wreath product $A\,wr B$ of groups A and B is a semidirect product $G = R \rtimes B$ where $R = \prod_{b\in B}^{\times} A^b$ is the direct product of copies of A, numbered by elements of B and B acts on R by shifting indices. Instead of A^e we write A, and $A^b = b^{-1}A\,b$. Every element $g \in G$ can be uniquely written as $g = bw$ where $b \in B$ and w is a product of commuting factors a^b, where $b \in B$, $a^b = b^{-1}a\,b$, $\ ba \cdot b_1 a_1 = bb_1 a^{b_1} a_1$.

Now we give a criterion for a restricted wreath product of groups to be an $FNCC$-group.

Theorem 3.9. *A restricted wreath product $A\,wr\,B$ is an $FNCC$-group if and only if A is an $FNCC$-group and B is finite.*

Proof. Let $G = A\,wr\,B = (\prod_{b\in B}^{\times} A^b) \rtimes B = R \rtimes B$ be an $FNCC$-group. In the restricted wreath product each element $r \in R$ has a finite support of the length $s(r)$, say. Moreover, the conjugate elements have supports of the same length. If G is an $FNCC$-group, then the lengths of possible supports have only finite number of values, which is possible only if the group B is finite. Then R is a subgroup of finite index in G and by Lemma 2.1(ii), R is an $FNCC$-group. Then A is an $FNCC$ group as an image of R.

The converse implication follows from Theorem 3.8, because the group B acts by permutations on the subscripts of direct factors in R. □

References

[1] R. G. Burns and Y. Medvedev, Group laws implying virtual nilpotence, *J. Aust. Math. Soc.* **74** (2003), 295-312.

[2] G. Cutolo, H. Smith and J. Wiegold, Groups covered by conjugates of proper subgroups, *J. Algebra* **293** (2005), 261-268.

[3] Yu.M. Gorchakov, *Groups with finite classes of conjugate elements*, Nauka, Moscow, 1978 (in Russian).

[4] Yu. N. Gorchinskii, Groups with a finie number of conjugacy classes, *Mat. Sbornik N.S.* **31** (1952), 167-182.

[5] Yu. N. Gorchinskii, Periodic groups with a finie number of conjugacy classes, *Mat. Sbornik N.S.* **31** (1952), 209-216.

[6] B. Hartley, *Simple locally finite groups*, in: *Finite and Locally Finite Groups, Istanbul, 1994*, NATO Adv. Sci. Inst. Ser.C Math. Phys. Sci., **471**, Kluwer Academic, Dordrecht, 1995, 1-44.

[7] I.N. Herstein, *Noncommutative rings*, MAA, 1968.

[8] M. Herzog, P. Longobardi and M. Maj, On generalized FC-groups, *J. Group Theory* **11** (1) (2008), 105–117.

[9] G. Higman, B. H. Neumann and H. Neumann, Embedding theorems for groups. *J. London Math. Soc.* **24** (1949). 247–254.

[10] E. I. Khukhro and V. D. Mazurov (eds.), *Unsolved problems in group theory*, The Kourovka Notebook, no. 17, Novosibirsk, 2010.

[11] J. Krempa, On finite generation of unit groups for group rings, in: *Groups '93 Galway/St. Andrews,* Vol. 2, London Math. Soc. Lecture Note Ser., **212**, Cambridge Univ. Press, Cambridge, 1995, 352–367.

[12] O. Macedońska and W. Tomaszewski, On Engel and positive laws, in: *Groups St Andrews 2009 in Bath,* Vol. 2, London Math. Soc. Lecture Note Ser., **388**, Cambridge Univ. Press, Cambridge, 2011, 461–472.

[13] A. Yu. Ol'shanskii, *Geometry of defining relations in groups*, Kluwer Academic Publishers Group, 1991.

[14] D.V. Osin, Small cancellations over relatively hyperbolic groups and embedding theorems, *Annals of Math.* (2) **172**(1) (2010), 1–39.

[15] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups* Part I, Ergebnisse der Mathematik und ihrer Grenzgebiete v.62, Springer-Verlag, New York Heidelberg Berlin, 1972.

[16] D.J.S. Robinson, *A course in the theory of groups*, Springer-Verlag, New York Heilderberg Berlin, 1982.

[17] M. J. Tomkinson. *FC-groups*, Pitman, Boston MA, 1984.

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